Index notation - Summary

Definition

For any number a and positive integer n,

$$a^n = \underbrace{a \times a \times a \times a \ldots \times a}_{n \text{ factors}}$$

By convention $a^1 = a$ and $a^0 = 1$ $(a \neq 0)$

$(0^0$ *does not exist*$)$

Examples:

"4 squared" = $4^2 = 4 \times 4 = 16$

"2 cubed" = $2^3 = 2 \times 2 \times 2 = 8$

$10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100000$

$34^1 = 34$ and $7.3^0 = 1$

Rules of operations

Rule 1: $a^n \times a^p = a^{n+p}$

Rule 2: $a^n \div a^p = \dfrac{a^n}{a^p} = a^{n-p}$

Rule 3: $\left(a^n\right)^p = a^{n \times p}$

Indices and fractions

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Examples and explanations:

$$7^2 \times 7^6 = \underbrace{7 \times 7}_{7^2} \times \underbrace{7 \times 7 \times 7 \times 7 \times 7 \times 7}_{7^6} = 7^8$$

$$\frac{8^9}{8^5} = \frac{8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8}{8 \times 8 \times 8 \times 8 \times 8}$$ and by simplifying:

$$\frac{8^9}{8^5} = \frac{\cancel{8} \times \cancel{8} \times \cancel{8} \times \cancel{8} \times \cancel{8} \times \boxed{8 \times 8 \times 8 \times 8}}{\cancel{8} \times \cancel{8} \times \cancel{8} \times \cancel{8} \times \cancel{8}} = 8^4$$

$$\left(5^3\right)^4 = 5^3 \times 5^3 \times 5^3 \times 5^3 = 5^{3+3+3+3} \text{ (rule 1)} = 5^{12}$$

Negative indices

Part 1: understanding what RECIPROCAL numbers are.

Two numbers are reciprocal if their product ($\times$) is equal to 1

Examples: 2 and $\frac{1}{2}$ are reciprocal because $2 \times \frac{1}{2} = 1$

$\frac{2}{3}$ and $\frac{3}{2}$ are reciprocal because $\frac{2}{3} \times \frac{3}{2} = \frac{6}{6} = 1$

- How do you work out the reciprocal of a number?

The reciprocal of any number $a \neq 0$ is $\frac{1}{a}$

The reciprocal of any fraction $\frac{a}{b}$ is $\frac{b}{a}$

Examples: The reciprocal of 6 is $\frac{1}{6}$, the reciprocal of 5.6 is $\frac{1}{5.6}$

The reciprocal of $\frac{4}{5}$ is $\frac{5}{4}$.

Part 2: Reciprocals and negative indices

According to the rule 1, $a^1 \times a^{-1} = a^{1+-1} = a^0 = 1$

$a^1 = a$, this tells us that a and a^{-1} are reciprocal

Conclusion: a^{-1} is the reciprocal of a : $a^{-1} = \frac{1}{a}$

Examples: $4^{-1} = \frac{1}{4} = 0.25$; $\left(\frac{3}{2}\right)^{-1} = \frac{2}{3}$

Indices of the form $\frac{1}{n}$

Part 1: understanding square root, cube root anf nth roots.

- $if \quad b=\sqrt{a} \quad then\ b^2=a$

  $\sqrt{16}=4 \ because\ 4^2=16$

- $if \quad b=\sqrt[3]{a} \quad then \quad b^3=a$

  $\sqrt[3]{8}=2 \quad because \quad 2\times2\times2=2^3=8$

  $\sqrt[3]{27}=3 \ because\ 3\times3\times3=3^3=27$

- We can extend this to any number n:

$$b=\sqrt[n]{a} \quad means\ that \quad b^n=a$$

$\sqrt[5]{100000}=10 \quad because \quad 10^5=100000$

$\sqrt[5]{32}=2 \quad because \quad 2^5=32$

Part 2: Roots and indices

We want to find the value of $a^{\frac{1}{n}}$.Let's call $b=a^{\frac{1}{n}}$.

According the rule 3, $b^n=\left(a^{\frac{1}{n}}\right)^n=a^{\frac{1}{n}\times n}=a^1=a$

The number b is such that $b^n=a \ and$ we have just established that it means $b=\sqrt[n]{a}$.

$$a^{\frac{1}{n}}=\sqrt[n]{a}$$

in particular $a^{\frac{1}{2}}=\sqrt{a}$ and $a^{\frac{1}{3}}=\sqrt[3]{a}$

B

Examples:

$16^{\frac{1}{2}}=\sqrt{16}=4 \qquad 7^{\frac{1}{2}}=\sqrt{7}$

$27^{\frac{1}{3}}=\sqrt[3]{27}=3 \qquad 64^{\frac{1}{6}}=\sqrt[6]{64}=2 \ because\ 2^6=64$

$100000000^{\frac{1}{8}}=\sqrt[8]{100000000}=10 \ because\ 10^8=100000000$

Fractional indices

Part 1: Positive fractional indices

Any fraction $\frac{n}{p}$ can be considered as the product of $n \ and \ \frac{1}{p}$

$$n\times\frac{1}{p}=\frac{1}{p}\times n=\frac{n}{p}$$

Considering this and rule 3 of operations, we have

$$a^{\frac{n}{p}}=a^{\frac{1}{p}\times n}=(a^{\frac{1}{p}})^n=\left(\sqrt[p]{a}\right)^n$$

$$a^{\frac{n}{p}}=\left(\sqrt[p]{a}\right)^n$$

A

Examples:

- $16^{\frac{3}{2}}$ means I have to square root 16 (power $\frac{1}{2}$)

  then cube the answer (power 3)

$16^{\frac{3}{2}}=(16^{\frac{1}{2}})^3=\sqrt{16}^3=4^3=64$

- $125^{\frac{2}{3}}=(125^{\frac{1}{3}})^2=\sqrt[3]{125}^2=5^2=25$

Part 2: Negative fractional indices

$a^{-\frac{n}{p}}$ is the reciprocal of $a^{\frac{n}{p}}$

$$a^{-\frac{n}{p}}=a^{-1\times n\times\frac{1}{p}}=\frac{1}{\left(\sqrt[p]{a}\right)^n}$$

A*

Examples:

- $16^{-\frac{5}{4}}$ means that you have to find successively

  the 4th *root*, the power 5, then the reciprocal

$16^{-\frac{5}{4}}=\frac{1}{\sqrt[4]{16}^5}=\frac{1}{2^5}=\frac{1}{32}$

- $\left(\frac{25}{36}\right)^{-\frac{3}{2}}$: $\sqrt{...}$, *cube, reciprocal*

$$\frac{25}{36}\xrightarrow{\sqrt{...}}\frac{5}{6}\xrightarrow{...^3}\frac{125}{196}\xrightarrow{reciprocal}\frac{196}{125}$$