Relative position – exam questions

Question 1: June 2006 – Q4

The unit vectors **i** and **j** are directed due east and due north respectively.

Two cyclists, Aazar and Ben, are cycling on straight horizontal roads with constant velocities of $(6\mathbf{i} + 12\mathbf{j})\,\text{km h}^{-1}$ and $(12\mathbf{i} - 8\mathbf{j})\,\text{km h}^{-1}$ respectively. Initially, Aazar and Ben have position vectors $(5\mathbf{i} - \mathbf{j})$ km and $(18\mathbf{i} + 5\mathbf{j})$ km respectively, relative to a fixed origin.

(a) Find, as a vector in terms of **i** and **j**, the velocity of Ben relative to Aazar. *(2 marks)*

(b) The position vector of Ben relative to Aazar at time t hours after they start is **r** km.

Show that

$$\mathbf{r} = (13 + 6t)\mathbf{i} + (6 - 20t)\mathbf{j}$$ *(4 marks)*

(c) Find the value of t when Aazar and Ben are closest together. *(6 marks)*

(d) Find the closest distance between Aazar and Ben. *(2 marks)*

Question 2: June 2007 – Q2

The unit vectors **i**, **j** and **k** are directed due east, due north and vertically upwards respectively.

Two helicopters, A and B, are flying with constant velocities of $(20\mathbf{i} - 10\mathbf{j} + 20\mathbf{k})\,\text{m s}^{-1}$ and $(30\mathbf{i} + 10\mathbf{j} + 10\mathbf{k})\,\text{m s}^{-1}$ respectively. At noon, the position vectors of A and B relative to a fixed origin, O, are $(8000\mathbf{i} + 1500\mathbf{j} + 3000\mathbf{k})$ m and $(2000\mathbf{i} + 500\mathbf{j} + 1000\mathbf{k})$ m respectively.

(a) Write down the velocity of A relative to B. *(2 marks)*

(b) Find the position vector of A relative to B at time t seconds after noon. *(3 marks)*

(c) Find the value of t when A and B are closest together. *(5 marks)*

Question 3: June 2008 – Q2

The unit vectors **i** and **j** are directed due east and due north respectively.

Two runners, Albina and Brian, are running on level parkland with constant velocities of $(5\mathbf{i} - \mathbf{j})\,\text{m s}^{-1}$ and $(3\mathbf{i} + 4\mathbf{j})\,\text{m s}^{-1}$ respectively. Initially, the position vectors of Albina and Brian are $(-60\mathbf{i} + 160\mathbf{j})$ m and $(40\mathbf{i} - 90\mathbf{j})$ m respectively, relative to a fixed origin in the parkland.

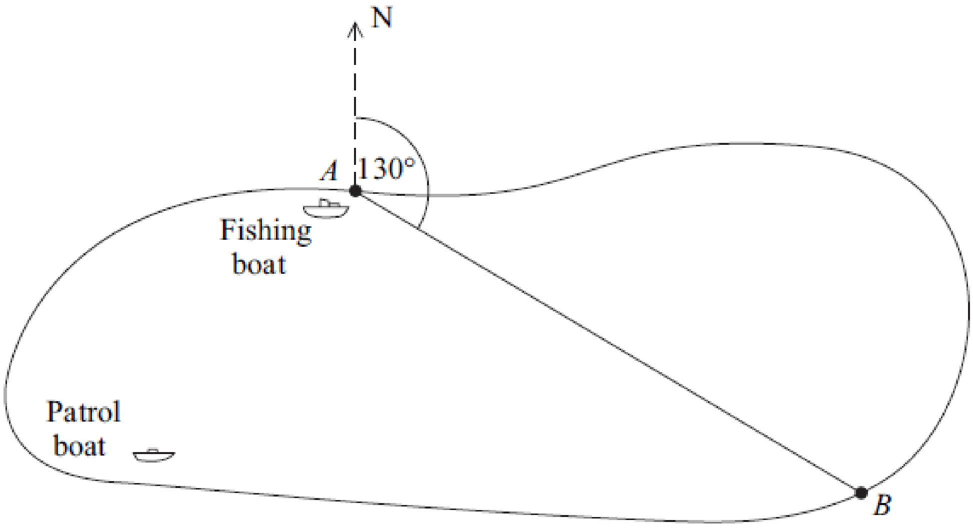
(a) Write down the velocity of Brian relative to Albina. *(2 marks)*

(b) Find the position vector of Brian relative to Albina t seconds after they leave their initial positions. *(3 marks)*

(c) Hence determine whether Albina and Brian will collide if they continue running with the same velocities. *(3 marks)*

Question 4: June 2009 – Q3

A fishing boat is travelling between two ports, A and B, on the shore of a lake. The bearing of B from A is 130°. The fishing boat leaves A and travels directly towards B with speed $2\,\text{m}\,\text{s}^{-1}$. A patrol boat on the lake is travelling with speed $4\,\text{m}\,\text{s}^{-1}$ on a bearing of 040°.



(a) Find the velocity of the fishing boat relative to the patrol boat, giving your answer as a speed together with a bearing. *(5 marks)*

(b) When the patrol boat is 1500 m due west of the fishing boat, it changes direction in order to intercept the fishing boat in the shortest possible time.

(i) Find the bearing on which the patrol boat should travel in order to intercept the fishing boat. *(4 marks)*

(ii) Given that the patrol boat intercepts the fishing boat before it reaches B, find the time, in seconds, that it takes the patrol boat to intercept the fishing boat after changing direction. *(4 marks)*

(iii) State a modelling assumption necessary for answering this question, other than the boats being particles. *(1 mark)*

Question 5: June 2010 – Q4

The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are directed east, north and vertically upwards respectively.

At time $t = 0$, the position vectors of two small aeroplanes, A and B, relative to a fixed origin O are $(-60\mathbf{i} + 30\mathbf{k})$ km and $(-40\mathbf{i} + 10\mathbf{j} - 10\mathbf{k})$ km respectively.

The aeroplane A is flying with constant velocity $(250\mathbf{i} + 50\mathbf{j} - 100\mathbf{k})\,\text{km}\,\text{h}^{-1}$ and the aeroplane B is flying with constant velocity $(200\mathbf{i} + 25\mathbf{j} + 50\mathbf{k})\,\text{km}\,\text{h}^{-1}$.

(a) Write down the position vectors of A and B at time t hours. *(3 marks)*

(b) Show that the position vector of A relative to B at time t hours is $((-20 + 50t)\mathbf{i} + (-10 + 25t)\mathbf{j} + (40 - 150t)\mathbf{k})$ km. *(2 marks)*

(c) Show that A and B do not collide. *(4 marks)*

(d) Find the value of t when A and B are closest together. *(6 marks)*

Question 6: June 2011 – Q4

The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are directed due east, due north and vertically upwards respectively.

A helicopter, A, is travelling in the direction of the vector $-2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ with constant speed $140\ \text{km h}^{-1}$. Another helicopter, B, is travelling in the direction of the vector $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ with constant speed $60\ \text{km h}^{-1}$.

(a) Find the velocity of A relative to B. *(5 marks)*

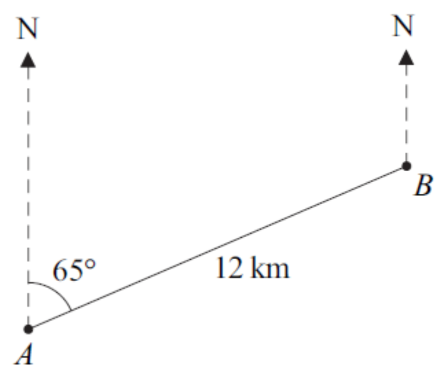
(b) Initially, the position vectors of A and B are $(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$ km and $(-3\mathbf{i} + 6\mathbf{j} + 3\mathbf{k})$ km respectively, relative to a fixed origin.

Write down the position vector of A relative to B, t hours after they leave their initial positions. *(2 marks)*

(c) Find the distance between A and B when they are closest together. *(8 marks)*

Question 7: June 2012 – Q6

At noon, two ships, A and B, are a distance of 12 km apart, with B on a bearing of 065° from A. The ship B travels due north at a constant speed of $10\ \text{km h}^{-1}$. The ship A travels at a constant speed of $18\ \text{km h}^{-1}$.



(a) Find the direction in which A should travel in order to intercept B. Give your answer as a bearing. *(4 marks)*

(b) In fact, the ship A actually travels on a bearing of 065°.

(i) Find the distance between the ships when they are closest together. *(7 marks)*

(ii) Find the time when the ships are closest together. *(3 marks)*

Relative position – exam questions - MS

Question 1: June 2006 – Q4

4(a)	${}_A\mathbf{v}_B = (12\mathbf{i}-8\mathbf{j})-(6\mathbf{i}+12\mathbf{j})$	M1	
	$=6\mathbf{i}-20\mathbf{j}$	A1	2
(b)	${}_A\mathbf{r}_B = \mathbf{r}_0 + {}_A\mathbf{v}_B t$	M1A1	
	${}_A\mathbf{r}_B = (18\mathbf{i}+5\mathbf{j})-(5\mathbf{i}-\mathbf{j})+(6\mathbf{i}-20\mathbf{j})t$	A1F	
	${}_A\mathbf{r}_B = (13+6t)\mathbf{i}+(6-20t)\mathbf{j}$	A1	4
	Alternative		
	$\mathbf{r}_A = 5\mathbf{i}-\mathbf{j}+(6\mathbf{i}+12\mathbf{j})t$		
	$\mathbf{r}_B = 18\mathbf{i}+5\mathbf{j}+(12\mathbf{i}-8\mathbf{j})t$	M1A1	
	${}_A\mathbf{r}_B = 18\mathbf{i}+5\mathbf{j}+(12\mathbf{i}-8\mathbf{j})t - [5\mathbf{i}-\mathbf{j}+(6\mathbf{i}+12\mathbf{j})t]$	A1	
	${}_A\mathbf{r}_B = (13+6t)\mathbf{i}+(6-20t)\mathbf{j}$	A1F	
(c)	$s^2=(13+6t)^2+(6-20t)^2$	M1A1F	
	A and B are closest when $\frac{ds}{dt}=0$ or	M1	
	$\frac{ds^2}{dt}=0$		
	$2s\frac{ds}{dt}=2(13+6t)6-2(6-20t)20=0$	M1 A1	
	$t=0.0963$	A1F	6
	$\left(\text{or } 0.096 \text{ or } \frac{21}{218}\right)$		

Question 2: June 2007 – Q2

2 (a)	${}_B\mathbf{v}_A = \mathbf{v}_A - \mathbf{v}_B$		
	$=(20\mathbf{i}-10\mathbf{j}+20\mathbf{k})-(30\mathbf{i}+10\mathbf{j}+10\mathbf{k})$	M1A1	2
	$=-10\mathbf{i}-20\mathbf{j}+10\mathbf{k}$		
(b)	${}_B\mathbf{r}_{0A} = (8000\mathbf{i}+1500\mathbf{j}+3000\mathbf{k}) -(2000\mathbf{i}+500\mathbf{j}+1000\mathbf{k})$	M1	
	$=6000\mathbf{i}+1000\mathbf{j}+2000\mathbf{k}$		
	${}_B\mathbf{r}_A = (6000\mathbf{i}+1000\mathbf{j}+2000\mathbf{k}) +(-10\mathbf{i}-20\mathbf{j}+10\mathbf{k})t$	M1 A1F	3
	${}_B\mathbf{r}_A = (6000-10t)\mathbf{i}+(1000-20t)\mathbf{j} +(2000+10t)\mathbf{k}$		
(c)	$\lvert {}_B\mathbf{r}_A\rvert^2 = (6000-10t)^2+(1000-20t)^2 +(2000+10t)^2$	M1 A1F	
	The helicopters are closest when $\lvert {}_B\mathbf{r}_A\rvert^2$ is minimum.		
	$y=(6000-10t)^2+(1000-20t)^2 +(2000+10t)^2$		
	$\frac{dy}{dt}=2(-10)(6000-10t) +2(-20)(1000-20t) + 2(10)(2000+10t)=0$	m1 A1F	
	$t=100$	A1F	5
	Alternative:		
	$\begin{pmatrix}6000-10t\\1000-20t\\2000+10t\end{pmatrix}\cdot\begin{pmatrix}-10\\-20\\10\end{pmatrix}=0$	(M1) (A1F)	
	$-60000+100t-20000+400t +20000+100t=0$	(m1) (A1F)	
	$600t=60000$		
	$t=100$	(A1F)	(5)
	Total		**10**

Question 3: June 2008 – Q2

(a)	${}_Av_B = v_B - v_A$		
	$=(3i+4j)-(5i-j)$	M1	
	$=-2i+5j$	A1	2
(b)	${}_Ar_{0B}=(40i-90j)-(-60i+160j)$	M1	
	$=100i-250j$		
	${}_Ar_B=(100i-250j)+(-2i+5j)t$	m1 A1F	3
(c)	${}_Ar_B=(100-2t)i+(-250+5t)j$	M1	
	$(100-2t)=0 \iff t=50$	A1F	
	$(-250+5t)=0 \iff t=50$		
	$\therefore$ A and B would collide.	E1	3

Question 4: June 2009 – Q3

3(a)	${}_Pv_F=\sqrt{4^2+2^2}$	M1	
	$=4.47\text{ m s}^{-1}$ or $2\sqrt{5}\text{ ms}^{-1}$ or $\sqrt{20}\text{ ms}^{-1}$	A1	
	$\theta=\tan^{-1}\frac{2}{4}$	M1	
	$\theta=26.6°$	A1F	
	Bearing $=40°+180°-26.6°$ $=193°$	A1F	5
	Alternative:		
	Comp. due west $=4\sin 40°-2\sin 50°=1.04\text{ ms}^{-1}$	(M1)	
	Comp. due south $=2\cos 50°+4\cos 40°=4.35\text{ ms}^{-1}$		
	${}_Pv_F=\sqrt{1.04^2+4.35^2}=4.47\text{ ms}^{-1}$	(A1)	
	$\theta=\tan^{-1}\frac{1.04}{4.35}$ or $\tan^{-1}\frac{4.35}{1.04}$	(M1)	
	$\theta=13.4°$ or $76.6°$	(A1F)	
	Bearing $=13.4°+180°$ or $270°-76.6°$ $=193°$	(A1F)	
	Alternative:		
	Correct triangle	(M1)	
	${}_Pv_F=\sqrt{1.04^2+4.35^2}=4.47\text{ms}^{-1}$	(A1)	
	Rel. Vel. Triangle angle $26.6°$ or $63.4°$	(A1)	
	Bearing $=40°+180°-26.6°$ or $63.4°+40°+90°$	(M1)	
	$=193°$	(A1F)	
(b)(i)	$v_F=v_P+{}_Pv_F$		
	$\frac{\sin\alpha}{2}=\frac{\sin 140°}{4}$	M1A1	
	$\alpha=18.7°$	A1F	
	Bearing $=90°+18.7°$ $=109°$	A1F	4
	Alternative:		
	$2\sin 40°=4\sin\alpha$	(M1)	
	$\alpha=\sin^{-1}\left(\frac{1}{2}\sin 40°\right)$	(A1)	
	$\alpha=18.7°$	(A1F)	
	Bearing $=109°$	(A1F)	

(b)(ii)	$\beta = 180° - (140° + 18.7°)$ $= 21.3°$	B1F	
	$\dfrac{{}_P v_F}{\sin 21.3°} = \dfrac{4}{\sin 140°}$	M1	
	${}_P v_F = 2.2568\,\text{ms}^{-1}$	A1F	
	$t = \dfrac{1500}{2.2568}$ $= 665$ sec	A1F	4
	Alternative: ${}_F v_P = 4\cos 18.7 - 2\cos 40 = 2.2568$	(M1) (A2,1,0)	
	$t = \dfrac{1500}{2.2568} = 665$ sec	(A1F)	
(iii)	No cross wind, calm lake, instantaneous change of direction by the patrol boat	B1	1
	Total		**14**

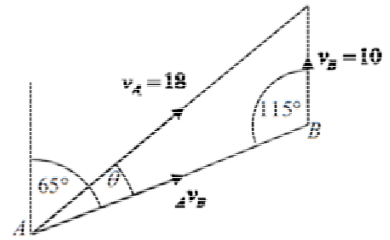
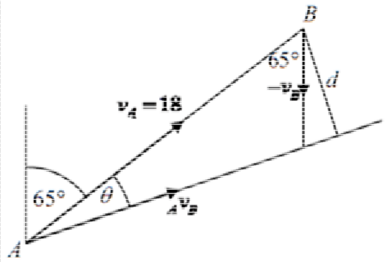
Question 5: June 2010 – Q4

(a)	$r_A = (-60\mathbf{i} + 30\mathbf{k}) + (250\mathbf{i} + 50\mathbf{j} - 100\mathbf{k})t$	M1	
	$r_B = (-40\mathbf{i} + 10\mathbf{j} - 10\mathbf{k}) + (200\mathbf{i} + 25\mathbf{j} + 50\mathbf{k})t$	A1,2	3
(b)	${}_B r_A = [\,(-60\mathbf{i} + 30\mathbf{k}) + (250\mathbf{i} + 50\mathbf{j} - 100\mathbf{k})t\,] -$ $[(-40\mathbf{i} + 10\mathbf{j} - 10\mathbf{k}) + (200\mathbf{i} + 25\mathbf{j} + 50\mathbf{k})t]$	M1	
	${}_B r_A = (-20 + 50t)\mathbf{i} + (-10 + 25t)\mathbf{j} + (40 - 150t)\mathbf{k}$	A1	2
(c)	For collision $(-20 + 50t)\mathbf{i} + (-10 + 25t)\mathbf{j} + (40 - 150t)\mathbf{k} = 0$	M1	
	$-20 + 50t = 0 \;\Rightarrow\; t = \frac{2}{5}$ $-10 + 25t = 0 \;\Rightarrow\; t = \frac{2}{5}$ $40 - 150t = 0 \;\Rightarrow\; t = \frac{4}{15}$	m1 A1F	
	The relative position vector cannot be zero. Therefore A and B do not collide	E1	4
(d)	$S^2 = (-20 + 50t)^2 + (-10 + 25t)^2 + (40 - 150t)^2$	M1A1	
	For minimum S $\dfrac{dS^2}{dt} = 100(-20 + 50t) + 50(-10 + 25t) -$ $300(40 - 150t) = 0$	M1 A1F	
	$51250t - 14500 = 0$ $t = 0.283$	m1 A1F	6
	Total		15

Question 6: June 2011 – Q4

(a)	$u_A = \dfrac{(-2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})140}{\sqrt{(2)^2 + (3)^2 + (6)^2}} = -40\mathbf{i} + 60\mathbf{j} + 120\mathbf{k}$	M1 A1	
	$u_B = \dfrac{(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})60}{\sqrt{(2)^2 + (1)^2 + (2)^2}} = 40\mathbf{i} - 20\mathbf{j} + 40\mathbf{k}$	A1	
	${}_A u_B = (-40\mathbf{i} + 60\mathbf{j} + 120\mathbf{k}) - (40\mathbf{i} - 20\mathbf{j} + 40\mathbf{k})$ $= -80\mathbf{i} + 80\mathbf{j} + 80\mathbf{k}$	M1 A1F	5
(b)	${}_A r_B = (4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) - (-3\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}) +$ $t(-80\mathbf{i} + 80\mathbf{j} + 80k)$ or $(7\mathbf{i} - 8\mathbf{j}) + t(-80\mathbf{i} + 80\mathbf{j} + 80\mathbf{k})$	M1 A1F	2
(c)	${}_A r_B = (7 - 80t)\mathbf{i} + (-8 + 80t)\mathbf{j} + (80t)\mathbf{k}$	B1F	
	$s^2 = (7 - 80t)^2 + (-8 + 80t)^2 + (80t)^2$	B1F	
	$2s\dfrac{ds}{dt} = 2(7 - 80t)(-80) + 2(-8 + 80t)(80) +$ $2(80t)(80) = 0$	M1 A1F	
	$240t = 15$	m1	
	$t = 0.0625$ or $\dfrac{1}{16}$	A1F	
	$s^2 = (7 - 80 \times 0.0625)^2 + (-8 + 80 \times 0.0625)^2 +$ $(80 \times 0.0625)^2$	M1	
	$s = 6.16$ km or $\sqrt{38}$ km	A1F	8
			15

Question 7: June 2012 – Q6

6(a)		B1	
	$\dfrac{\sin\theta}{10} = \dfrac{\sin 115°}{18}$	M1	
	$\theta = 30.2°$	A1	
	Bearing $= 035°$	A1	4
(b)(i)		B1	
	${}_A v_B^{\,2} = 18^2 + 10^2 - 2(18)(10)\cos 65°$	M1	
	${}_A v_B = 16.4881\ \text{ms}^{-1}$	A1	
	$\dfrac{\sin 65°}{16.4881} = \dfrac{\sin\theta}{10}$	M1	
	$\theta = 33.3446°$	A1F	
	$d = 12 \times \sin 33.3446°$	m1	
	$d = 6.60$ km	A1F	7
(ii)	$t = \dfrac{12 \times \cos 33.3446°}{16.4881} = 0.607987$ hours $(\ = 36.5$ min$)$	M1 A1F A1F	3
	Total		**14**