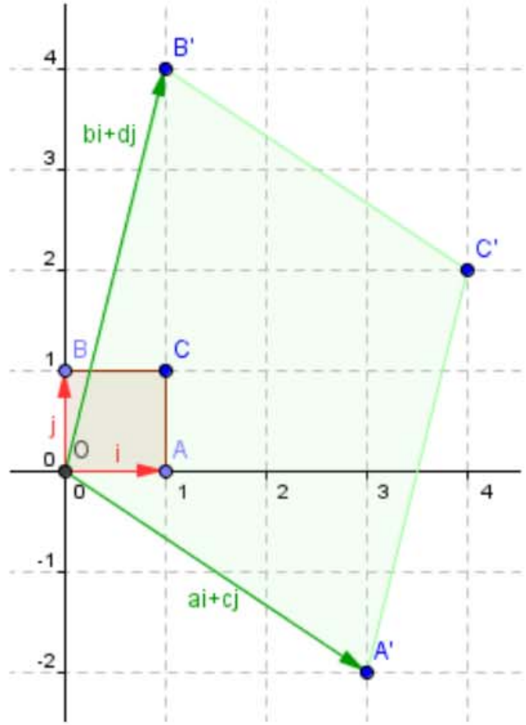
Manipulating determinants

Specifications:

Determinants

Second order and third order determinants, and their manipulation.	Including the use of the result $\det(\mathbf{AB}) = \det\mathbf{A}\det\mathbf{B}$, but a general treatment of products is not required.
Factorisation of determinants.	Using row and/or column operations or other suitable methods.

In the matrix algebra chapter, we have defined the determinant of a matrix by considering the image of the base vectors through the matrix transformation:



- For a 2×2 matrix, $\mathbf{M}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

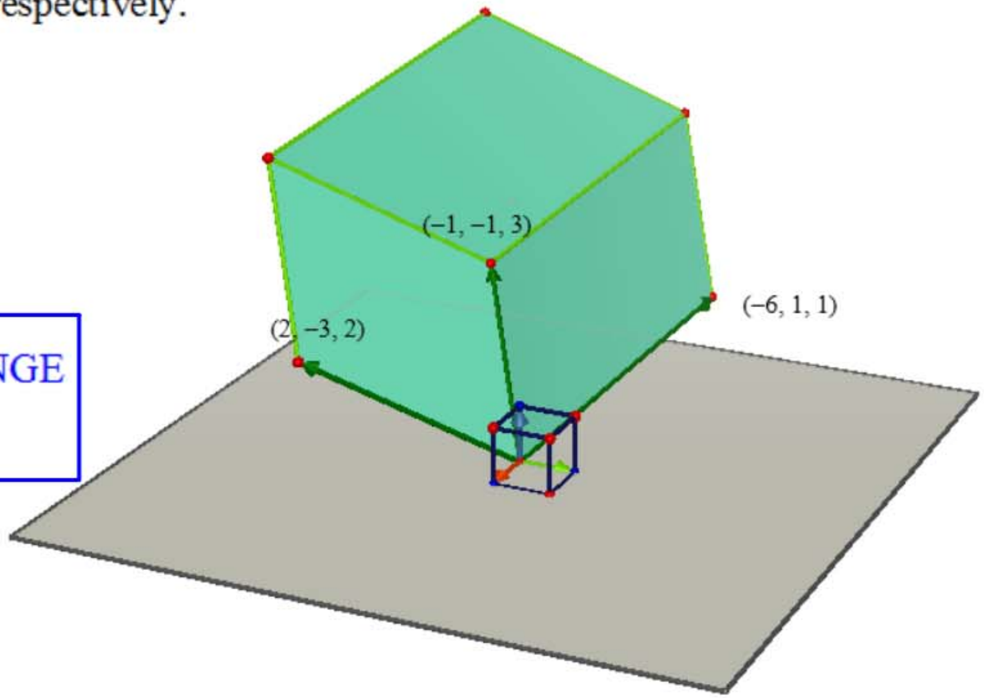
  $\mathbf{i}$ *and* $\mathbf{j}$ *are mapped onto* $\mathbf{a}=\begin{pmatrix} a \\ c \end{pmatrix}$ *and* $\mathbf{b}=\begin{pmatrix} b \\ d \end{pmatrix}$ respectively.

  $$\det(\mathbf{M})=|\mathbf{a}\times\mathbf{b}|=ad-bc$$

- For a 3×3 matrix, $\mathbf{M}=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$,

  $\mathbf{i},\mathbf{j}$ *and* $\mathbf{k}$ *are mapped onto* $\mathbf{a}=\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \mathbf{b}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ *and* $\mathbf{c}=\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ respectively.

  $$\det(\mathbf{M})=\mathbf{a}.(\mathbf{b}\times\mathbf{c})$$



Property of the triple scalar product: CYCLIC INTERCHANGE

$$\mathbf{a}.(\mathbf{b}\times\mathbf{c})=\mathbf{b}.(\mathbf{c}\times\mathbf{a})=\mathbf{c}.(\mathbf{a}\times\mathbf{b})$$

Reminder

Property 1: For two matrices $\mathbf{A}$ and $\mathbf{B}$ so that $\mathbf{AB}$ exists,

$$\det(\mathbf{AB}) = \det(\mathbf{A}) \times \det(\mathbf{B})$$

Vocabulary: $\mathbf{A}$ is SINGULAR when $\det(\mathbf{A}) = 0$

(When A is singular, A^{-1} *does not exist*)

Property 2: For a matrix $\mathbf{A}$ of order n and a scaler λ,

$$\det(\lambda\mathbf{A}) = \lambda^n \det(\mathbf{A})$$

In particular:

$$\det(\lambda\mathbf{I}_2) = \lambda^2 \quad \textit{and} \quad \det(\lambda\mathbf{I}_3) = \lambda^3$$

where $\mathbf{I}_2$ *and* $\mathbf{I}_3$ are the identity matrices of order 2 and 3 respectively.

Exercises:

- The matrices $\mathbf{P}$ and $\mathbf{Q}$ are defined in terms of the constant k by

$$\mathbf{P} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & k \\ 5 & 3 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} 5 & 4 & 1 \\ 3 & k & -1 \\ 7 & 3 & 2 \end{bmatrix}$$

(a) Express $\det \mathbf{P}$ and $\det \mathbf{Q}$ in terms of k. *(3 marks)*

(b) Given that $\det(\mathbf{PQ}) = 16$, find the two possible values of k. *(4 marks)*

- The 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ are such that

$$\mathbf{AB} = \begin{bmatrix} 9 & 1 \\ 7 & 13 \end{bmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{bmatrix} 14 & 2 \\ 1 & 8 \end{bmatrix}$$

Without finding $\mathbf{A}$ and $\mathbf{B}$:

(a) find the value of $\det \mathbf{B}$, given that $\det \mathbf{A} = 10$; *(3 marks)*

- The matrix $\mathbf{A} = \begin{bmatrix} k & 1 & 2 \\ 2 & k & 1 \\ 1 & 2 & k \end{bmatrix}$, where k is a real constant.

(i) Find $\det \mathbf{A}$ in terms of k. *(2 marks)*

(ii) In the case when $\mathbf{A}$ is singular, find the integer value of k and show that there are no other possible real values of k. *(3 marks)*

- $\mathbf{P} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & t & -2 \\ 3 & 2 & 1 \end{bmatrix}$

(a) Find the value of t for which $\mathbf{P}$ is singular. *(2 marks)*

- The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & t \end{bmatrix}$$

(a) Find, in terms of t, the matrices:

(i) $\mathbf{AB}$; *(3 marks)*

(ii) $\mathbf{BA}$. *(2 marks)*

(b) Explain why $\mathbf{AB}$ is singular for all values of t. *(1 mark)*

Properties of the determinant

1) $$|\mathbf{M}| = |\mathbf{M}^{\mathrm{T}}|$$

A consequence of this rule is that anything you can prove for columns of a determinant must also be true for rows.

2) **Adding or subtracting any multiple of a row (or column) to another row (or column) does not affect the determinant**

Consider the three vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ *and* $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

We have established that $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$ is the determinant of the matrix $M = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

To prove this property, let's work out the determinant of this three vectors after replacing $\mathbf{c}$ by $\mathbf{c} + \lambda \mathbf{b}$

$$\mathbf{a}.(\mathbf{b} \times (\mathbf{c} + \lambda \mathbf{b})) =$$

Application:

Work out $\Delta = \begin{vmatrix} a & -a & -a \\ 4 & 8 & -4 \\ x & 4x & 2x \end{vmatrix}$

3) Interchanging two rows (or columns) of a matrix changes the sign of the determinant

Proof: The determinanat of M is $\mathbf{a}.(\mathbf{b}\times\mathbf{c})$ $\left(=\mathbf{b}.(\mathbf{c}\times\mathbf{a})=\mathbf{c}.(\mathbf{a}\times\mathbf{b})\right.$ (*cyclic* interchange))

If you swap the first and the second column,

the new determinant is $\mathbf{b}.(\mathbf{a}\times\mathbf{c})=$

4) Multiplying a row (or column) of a matrix by a scalar multiplies the determinant by that scalar

Proof:

Consequence:

To factorise a determinant, use row (or column) operations to obtain a row (or column) of elements with a common factor.

$$\textit{Work out } \Delta=\begin{vmatrix}4 & -3 & 1\\ 2 & -5 & 5\\ 1 & -1 & 2\end{vmatrix}$$

One purpose of these properties is to re-arrange a determinant by combining rows and columns in order to obtain as many elements as possible
either equal to 0
or with a common factor

In particular, it is very easy to work out the determinant of the following matrices:

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} = a\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = ab_2c_3 - ac_2b_3$$

Triangular matrices:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix} = a_1b_2c_3$$

Example:

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 3 & 6 \end{vmatrix}\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & -1 & 0 \end{vmatrix}\begin{matrix} r_1 \\ r_2 - r_1 \\ r_3 - 2r_1 \end{matrix} = 1\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} = 2$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Examples:

Multiply out (a) $\begin{vmatrix} -1 & 2 & -3 \\ 3 & -1 & 7 \\ -1 & 2 & -3 \end{vmatrix}$, (b) $\begin{vmatrix} 4 & 8 & 16 \\ -1 & -2 & 6 \\ 3 & 1 & 4 \end{vmatrix}$ (c) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Your turn:

1. Evaluate (a) $\begin{vmatrix} 5 & 7 & 9 \\ 4 & 5 & 7 \\ 4 & 6 & 8 \end{vmatrix}$, (b) $\begin{vmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix}$.

2. Factorise $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$.

Exercises:

1) Find the values of each of these determinants.

(a) $\begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -4 \\ -3 & 1 & 4 \end{vmatrix}$, (b) $\begin{vmatrix} p & 2p \\ 3p & 4p \end{vmatrix}$, (c) $\begin{vmatrix} p & 2q & 3r \\ 2p & 3q & 4r \\ 3p & 4q & 6r \end{vmatrix}$

2. Express the following determinant as the product of four linear factors:

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}.$$

3. (a) Express the determinant $D = \begin{vmatrix} a^3+a^2 & a & 1 \\ b^3+b^2 & b & 1 \\ c^3+c^2 & c & 1 \end{vmatrix}$ as the product of four linear factors.

(b) Given that no two of a, b and c are equal and that $D = 0$, find the value of $a+b+c$.

4. Show that $\begin{vmatrix} k+4 & 5k+7 & k+1 \\ k+2 & 4k+7 & k \\ k+1 & 4k+5 & k-1 \end{vmatrix}$ has the same value for all values of k.

5. Factorise each of these determinants.

(a) $\begin{vmatrix} a & b & c \\ b+c & a+c & a+b \\ bc & ac & ab \end{vmatrix}$, (b) $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$.

6. The numbers a, b and c are all different and $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = 0$.

Show that $ab+bc+ca = 0$.

7. Solve the equation $\begin{vmatrix} 0 & x-1 & x^2-1 \\ 2x & x & (x+1)^2 \\ 1-x & 1 & 0 \end{vmatrix} = 0$.