Centre Number						Candidate Number				
Surname										
Other Names										
Candidate Signature										

For Examiner's Use	
Examiner's Initials	
Question	Mark
1	
2	
3	
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6	
7	
8	
TOTAL	

AQA

General Certificate of Education
Advanced Level Examination
June 2015

Mathematics MFP4

Unit Further Pure 4

Wednesday 20 May 2015 9.00 am to 10.30 am

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer each question in the space provided for that question. If you require extra space, use an AQA supplementary answer book; do **not** use the space provided for a different question.
- Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

Answer **all** questions.

Answer each question in the space provided for that question.

1 The points U, V and W have position vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ respectively relative to an origin O, where

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} a \\ 7 \\ -2 \end{bmatrix}$$

(a) Find $(\mathbf{u} \times \mathbf{v}).\mathbf{w}$ in terms of a.

[2 marks]

(b) Given that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent:

(i) find the value of a;

[1 mark]

(ii) express $\mathbf{u}$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

[3 marks]

2 The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such that $\mathbf{c} \times \mathbf{a} = 2\mathbf{i}$ and $\mathbf{b} \times \mathbf{a} = 3\mathbf{j}$.

Simplify $(\mathbf{a} + 2\mathbf{b} - 6\mathbf{c}) \times (\mathbf{a} - \mathbf{b} + 3\mathbf{c})$, giving your answer in the form $\lambda\mathbf{i} + \mu\mathbf{j}$.

[5 marks]

3 (a) Factorise completely the determinant

$$\begin{vmatrix} a & b-c & -bc \\ b & a-c & -ca \\ -c & a+b & ab \end{vmatrix}$$

[6 marks]

(b) Hence, or otherwise, find the values of a for which the equations

$$\begin{aligned} ax + \quad\quad\quad y - \;\; 6z &= 0 \\ 3x + (a-2)y - 2az &= 0 \\ -2x + (a+3)y + 3az &= 0 \end{aligned}$$

do not have a unique solution.

[3 marks]

4 (a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

[6 marks]

(b) Given that $\mathbf{U} = \begin{bmatrix} 4 & b \\ a & -2 \end{bmatrix}$ and $\mathbf{U}^{-1}\mathbf{M}\mathbf{U} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, find the values of a and b.

[3 marks]

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5 A system of equations is given by

$$2x - 11y - 3z = 1$$
$$5x - 10y - 4z = 6$$
$$9x - 17y - 7z = 11$$

(a) Find the solution of this system of equations, showing all your working.

[5 marks]

(b) Interpret this solution geometrically.

[1 mark]

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6 The line L has equation $\left(\mathbf{r} - \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) \times \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The plane Π contains the line L and the point $A(4,\ 1,\ -2)$.

(a) Show that A does not lie on the line L.

[1 mark]

(b) Find an equation of the plane Π, giving your answer in the form $\mathbf{r}.\mathbf{n} = c$.

[5 marks]

(c) The point D has coordinates $(8,\ -2,\ 6)$. Find the coordinates of the image of D after reflection in the plane Π.

[5 marks]

7 The matrix $\mathbf{A} = \begin{bmatrix} 3.4 & 2 \\ 1.2 & 1 \end{bmatrix}$ represents a transformation that is a shear S followed by a transformation T.

(a) The shear S is such that the image of the point $(1,\ 1)$ is $(5,\ -3)$ and the line $y = -x$ is a line of invariant points. Find the matrix that represents S.

[4 marks]

(b) (i) Hence find the matrix that represents the transformation T.

[4 marks]

(ii) Give a full description of the transformation T.

[2 marks]

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8 The linear transformation T is represented by the matrix $\mathbf{M} = \begin{bmatrix} 1 & 2 & k \\ 0 & 3 & 4 \\ -1 & 1 & -1 \end{bmatrix}$.

(a) In the case when $\mathbf{M}$ is a non-singular matrix:

(i) find the possible values of k;

[3 marks]

(ii) find $\mathbf{M}^{-1}$ in terms of k.

[5 marks]

(b) In the case when $k = 1$, the matrix $\mathbf{M}^{-1}$ is applied to a solid shape of volume $6\,\mathrm{cm}^3$. Find the volume of the image.

[3 marks]

(c) In the case when $k = 5$, verify that the image of every point under T lies in the plane $x - y + z = 0$.

[3 marks]

(d) Find the value of k for which T has a line of invariant points and obtain the Cartesian equations of this line.

[5 marks]

END OF QUESTIONS

AQA – Further pure 4 – Jun 2015 – Answers

Question 1:

Exam report

$$1)a)\ (\mathbf{u}\times\mathbf{v}).\mathbf{w}=\begin{vmatrix}1 & 3 & a\\ 2 & -4 & 7\\ 2 & 2 & -2\end{vmatrix}=\begin{vmatrix}-4 & 7\\ 2 & -2\end{vmatrix}-2\begin{vmatrix}3 & a\\ 2 & -2\end{vmatrix}+2\begin{vmatrix}3 & a\\ -4 & 7\end{vmatrix}$$

$$=-6-2\times(-6-2a)+2(21+4a)=12a+48$$

$b)\,i)$ Three vectors are linearly dependent means that $(\mathbf{u}\times\mathbf{v}).\mathbf{w}=0$

$$12a+48=0\ \ for\ \ a=-4$$

$ii)$ $\mathbf{u}=\alpha\mathbf{v}+\beta\mathbf{w}$ *where* α *and* β *have to be found* :

$$\begin{cases}1=3\alpha-4\beta\\ 2=-4\alpha+7\beta\\ 2=2\alpha-2\beta\end{cases}\Leftrightarrow\begin{cases}1=3\alpha-4\beta\\ 2=-4\alpha+7\beta\\ 1+\beta=\alpha\end{cases}\Leftrightarrow\begin{cases}1=3+3\beta-4\beta\\ 2=-4\alpha+7\beta\\ \alpha=1+\beta\end{cases}\Leftrightarrow\begin{cases}\beta=2\\ 2=-4\alpha+7\beta\\ \alpha=3\end{cases}$$

We use the second equation to check the consistency:

$$-4\times3+7\times2=-12+14=2$$

Conclusion : $\mathbf{u}=3\mathbf{v}+2\mathbf{w}$

Question 2:

Exam report

$(\mathbf{a}+2\mathbf{b}-6\mathbf{c})\times(\mathbf{a}-\mathbf{b}+3\mathbf{c})=$

$\mathbf{a}\times\mathbf{a}-\mathbf{a}\times\mathbf{b}+3(\mathbf{a}\times\mathbf{c})+2(\mathbf{b}\times\mathbf{a})-2(\mathbf{b}\times\mathbf{b})+6(\mathbf{b}\times\mathbf{c})-6(\mathbf{c}\times\mathbf{a})+6(\mathbf{c}\times\mathbf{b})-18(\mathbf{c}\times\mathbf{c})$

$=\mathbf{0}\quad+3\mathbf{j}\quad-6\mathbf{i}\quad+6\mathbf{j}\quad-\quad\mathbf{0}\quad\cancel{+6(\mathbf{b}\times\mathbf{c})}\quad-12\mathbf{i}\quad\cancel{-6(\mathbf{b}\times\mathbf{c})}\quad-\mathbf{0}$

$=-18\mathbf{i}+9\mathbf{j}$

Question 3:

Exam report

$$a)\begin{vmatrix}a & b-c & -bc\\ b & a-c & -ca\\ -c & a+b & ab\end{vmatrix}=\begin{vmatrix}a+b-c & b-c & -bc\\ a+b-c & a-c & -ca\\ a+b-c & a+b & ab\end{vmatrix}$$

$$=(a+b-c)\begin{vmatrix}1 & b-c & -bc\\ 1 & a-c & -ca\\ 1 & a+b & ab\end{vmatrix}=(a+b-c)\begin{vmatrix}1 & b-c & -bc\\ 0 & a-b & -ca+bc\\ 0 & a+c & ab+bc\end{vmatrix}$$

$$=(a+b-c)\big((a-b)(ab+bc)-(a+c)(bc-ac)\big)$$

$$=(a+b-c)\big(b(a+c)(a-b)+b(a+c)(a-b)\big)$$

$$=2b(a+c)(a-b)(a+b-c)$$

b) The matrix of this set of equations is $\begin{bmatrix}a & 1 & -6\\ 3 & a-2 & -2a\\ -2 & a+3 & 3a\end{bmatrix}$

It is the matrix of part a) with $a=a,\ c=2$ *and* $b=3$

The determinant of this matrix is then :$6(a+2)(a-3)(a+1)$

The equations do not have a unique solution when $a=-2$ *or* $a=3$ *or* $a=-1$

(the values that make the determinant equal to 0)

Question 4:	Exam report
$a)$ Let's solve the equation det($\mathbf{M}$-$\lambda\mathbf{I}$)=0 $\begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0 \Leftrightarrow (1-\lambda)(4-\lambda)+2=0$ $\Leftrightarrow \lambda^2-5\lambda+6=0 \Leftrightarrow \lambda=2 \text{ or } \lambda=3$ $for\ \lambda=2:\ (\mathbf{M}-2\lambda)\begin{pmatrix} x \\ y \end{pmatrix}=0 \Leftrightarrow \begin{cases} -x-y=0 \\ 2x+2y=0 \end{cases}$ An eigenvector for the eigenvalue λ=2 is $\mathbf{u}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $for\ \lambda=3:\ (\mathbf{M}-3\lambda)\begin{pmatrix} x \\ y \end{pmatrix}=0 \Leftrightarrow \begin{cases} -2x-y=0 \\ 2x+y=0 \end{cases}$ An eigenvector for the eigenvalue λ=3 is $\mathbf{v}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $b)\begin{pmatrix} 4 \\ a \end{pmatrix}$ is an eigenvector for the value 3, so it is proportional to **v**: $a=-8$ $\begin{pmatrix} b \\ -2 \end{pmatrix}$ is an eigenvector for the value 2, so it is proportional to **u**: $b=2$	

Question 5:	Exam report
$\begin{cases} 2x-11y-3z=1 \\ 5x-10y-4z=6 \\ 9x-17y-7z=11 \end{cases} \Leftrightarrow \begin{cases} 2x-11y-3z=1 & \\ 35y-7z=7 & (2L_2-5L_1) \\ 65y-13z=13 & (2L_3-9L_1) \end{cases}$ $\Leftrightarrow \begin{cases} 2x-11y-3z=1 \\ 5y-z=1 \\ 5y-z=1 \end{cases} \Leftrightarrow \begin{cases} 2x=1+11(\lambda)+3(-1-5\lambda) \\ y=\lambda \quad (arbitrary\ choice) \\ z=-1-5\lambda \end{cases}$ $\Leftrightarrow \begin{cases} x=-1-2\lambda \\ y=\lambda \\ z=-1-5\lambda \end{cases}$ The solution is the line going through $(-1,0,-1)$ with direction vector $\begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix}$ $b)$ This means that the three plans meet at one single line : SHEAF	

Question 6:	Exam report
a) $\begin{pmatrix}4\\1\\-2\end{pmatrix}-\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}2\\0\\-2\end{pmatrix}$ *and* $\begin{pmatrix}2\\0\\-2\end{pmatrix}\times\begin{pmatrix}3\\-1\\2\end{pmatrix}=\begin{pmatrix}-2\\-10\\-2\end{pmatrix}$ *not* $\begin{pmatrix}0\\0\\0\end{pmatrix}$ A does not belong to the line L. b) The plan Π contains the point $A(4,1,-2)$ and the point $B(2,1,0)$ The vector $\mathbf{u}=3\mathbf{i}-\mathbf{j}+2\mathbf{k}$ is a direction vector of the plan because L is includedin it. The vector $\overrightarrow{AB}\times\mathbf{u}$ is a normal vector to the plan Π. $\begin{pmatrix}-2\\0\\2\end{pmatrix}\times\begin{pmatrix}3\\-1\\2\end{pmatrix}=\begin{pmatrix}2\\10\\2\end{pmatrix}$ *Let's call* $\mathbf{n}=\begin{pmatrix}1\\5\\1\end{pmatrix}$ *a normal vector to* Π. An equation of the plan is $r.\begin{pmatrix}1\\5\\1\end{pmatrix}=\begin{pmatrix}4\\1\\-2\end{pmatrix}.\begin{pmatrix}1\\5\\1\end{pmatrix}\Leftrightarrow r.\begin{pmatrix}1\\5\\1\end{pmatrix}=7$ c) Let's call $H(x,y,z)$ the point on Π so that DH is the shortest distance from D to Π. H is such that $\begin{cases}x=8+\lambda\\y=-2+5\lambda\\z=6+\lambda\end{cases}$ *and H belongs to* Π *so* $x+5y+z=7\Leftrightarrow(8+\lambda)+5(-2+5\lambda)+(6+\lambda)=7$ $\Leftrightarrow 27\lambda=3\Leftrightarrow\lambda=\frac{1}{9}$ The image of D after the reflection is then $\begin{cases}x=8+2\lambda\\y=-2+10\lambda\\z=6+2\lambda\end{cases}\Leftrightarrow\begin{cases}x=8\frac{2}{9}\\y=-\frac{8}{9}\\z=6\frac{2}{9}\end{cases}$	

Question 7:	Exam report
a) $S=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ $S\times\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}5\\-3\end{pmatrix}$ *gives* $\begin{cases}a+b=5\\c+d=-3\end{cases}$ $S\times\begin{pmatrix}x\\-x\end{pmatrix}=\begin{pmatrix}ax-bx\\cx-dx\end{pmatrix}=\begin{pmatrix}x\\-x\end{pmatrix}$ *gives* $\begin{cases}a-b=1\\c-d=-1\end{cases}$ These 4 equations together lead to $a=3,\ b=2,\ c=-2,\ d=-1$ The matrix of the shear is $S=\begin{bmatrix}3&2\\-2&-1\end{bmatrix}$ b)i) $A=T\times S$ so $T=A\times S^{-1}$ $S^{-1}=\frac{1}{1}\begin{bmatrix}-1&-2\\2&3\end{bmatrix}$ *and* $A\times S^{-1}=\begin{bmatrix}3.4&2\\1.2&1\end{bmatrix}\times\begin{bmatrix}-1&-2\\2&3\end{bmatrix}=\begin{bmatrix}0.6&-0.8\\0.8&0.6\end{bmatrix}$ ii) T is the rotation around (0,0) by the angle 53.1^o *anticlockwise.* (*form* $\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$)	

Question 8:	Exam report

a) $M = \begin{bmatrix} 1 & 2 & k \\ 0 & 3 & 4 \\ -1 & 1 & -1 \end{bmatrix}$

i) $\det(M) = \begin{vmatrix} 1 & 2 & k \\ 0 & 3 & 4 \\ -1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} - \begin{vmatrix} 2 & k \\ 3 & 4 \end{vmatrix} = -7 - 8 + 3k = 3k - 15$

M is non-sigular when det(M) $\neq 0$, $k \in \mathbb{R}$ *but* $k \neq 5$

ii) $\text{cofactors(M)} = \begin{bmatrix} -7 & -4 & 3 \\ k+2 & k-1 & -3 \\ 8-3k & -4 & 3 \end{bmatrix}$ *so* $\mathbf{M}^{-1} = \frac{1}{3k-15}\begin{bmatrix} -7 & k+2 & 8-3k \\ -4 & k-1 & -4 \\ 3 & -3 & 3 \end{bmatrix}$

b) *when* $k = 1, \det(M^{-1}) = \frac{1}{\det(M)} = \frac{1}{-12}$

The volume of the image is $6 \times \frac{1}{12} = 0.5\text{cm}^3$

c) *When* $k = 5, M = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 4 \\ -1 & 1 & -1 \end{bmatrix}$ *and* $M \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x+2y+5z \\ 3y+4z \\ -x+y-z \end{pmatrix}$

Now let's work out $x' - y' + z' = (x+2y+5z) - (3y+4z) + (-x+y-z) =$

$= x - x + 2y - 3y + y + 5z - 4z - z = 0$ *for all* $x, y,$ *and* z.

d) $\text{M} = \begin{bmatrix} 1 & 2 & k \\ 0 & 3 & 4 \\ -1 & 1 & -1 \end{bmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ *gives* : $\begin{cases} x + 2y + kz = x \\ 3y + 4z = y \\ -x + y - z = z \end{cases} \Leftrightarrow \begin{cases} 2y + kz = 0 \\ 2y + 4z = 0 \\ -x + y - 2z = 0 \end{cases}$

This is consistent only for $k = 4$.

$\begin{cases} \text{if} \quad z = \lambda \\ y = -2\lambda \\ x = -4\lambda \end{cases}$ The line goes through (0,0,0) with direction vector $\begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix}$

Equation of the line: $\frac{x}{-4} = \frac{y}{-2} = z$

Grade boundaries							
Grade		A*	A	B	C	D	E
Mark	Max 75						