General Certificate of Education
Advanced Level Examination
June 2011

Mathematics MFP4

Unit Further Pure 4

Wednesday 22 June 2011 9.00 am to 10.30 am

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

1 The matrices **A** and **B** are given in terms of p by

$$\mathbf{A} = \begin{bmatrix} 1 & p & 4 \\ -3 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} p & 1 & 5 \\ 9 & p & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

(a) Find each of $\det \mathbf{A}$ and $\det \mathbf{B}$ in terms of p. *(3 marks)*

(b) Without finding **AB**, determine all values of p for which **AB** is singular. *(3 marks)*

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2 The plane transformation T is the composition of a reflection in the line $y = x \tan \alpha$ followed by an anticlockwise rotation about O through an angle β.

Determine the matrix which represents T, and hence describe T as a single transformation. *(6 marks)*

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3 Given the vectors $\mathbf{p} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 7 \\ -2 \\ 4 \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} 2 \\ 3 \\ t \end{bmatrix}$, where t is a scalar parameter, determine the value of t in each of the following cases:

(a) $\mathbf{p} \times \mathbf{q}$ is parallel to $\mathbf{r}$; *(3 marks)*

(b) **p**, **q** and **r** are linearly dependent. *(3 marks)*

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4 The system of equations S is given in terms of the real parameters a and b by

$$\begin{aligned} 2x + y + 3z &= a + 1 \\ 5x - 2y + (a+1)z &= 3 \\ ax + 2y + 4z &= b \end{aligned}$$

(a) Find the two values of a for which S does not have a unique solution. *(4 marks)*

(b) In the case when $a = 2$, determine the value of b for which S has infinitely many solutions. *(4 marks)*

5 (a) (i) Find the eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -2 & 8 \end{bmatrix}$. *(6 marks)*

(ii) Hence write down each of the matrices $\mathbf{U}$, $\mathbf{D}$ and $\mathbf{U}^{-1}$ such that $\mathbf{A} = \mathbf{UDU}^{-1}$, where $\mathbf{D}$ is a diagonal matrix. *(4 marks)*

(b) A 2×2 matrix $\mathbf{M}$ has distinct real eigenvalues λ and μ, with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

(i) By considering the diagonalised form of $\mathbf{M}$, determine the eigenvalues of $\mathbf{M}^3$. *(2 marks)*

(ii) Write down the eigenvectors of $\mathbf{M}^3$. *(1 mark)*

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6 (a) The transformation U of three-dimensional space is represented by the matrix

$$\begin{bmatrix} 1 & 4 & -3 \\ 2 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

(i) Write down a vector equation for the line L with cartesian equation

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}$$ *(2 marks)*

(ii) Find a vector equation for the image of L under U, and deduce that it is a line through the origin. *(4 marks)*

(b) The plane transformation V is represented by the matrix $\begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$.

L_1 is the line with equation $y = \frac{1}{2}x + k$, and L_2 is the image of L_1 under V.

(i) Find, in the form $y = mx + c$, the cartesian equation for L_2. *(4 marks)*

(ii) Deduce that L_2 is parallel to L_1 and find, in terms of k, the distance between these two lines. *(3 marks)*

Turn over ▶

7 Let $\Delta = \begin{vmatrix} n(n+1) & n+1 & -1 \\ 0 & 1 & n \\ 1 & -(n+1) & 1 \end{vmatrix}$.

(a) (i) Show that $(n^2 + n + 1)$ is a factor of Δ. *(2 marks)*

(ii) Hence, or otherwise, express Δ in factorised form. *(2 marks)*

(b) By expanding Δ directly, show that

$$\Delta = [n(n+1)]^2 + \mathrm{f}(n)$$

where $\mathrm{f}(n)$ can be expressed as the sum of two squares. *(2 marks)*

(c) Hence express the number 12 321 as the sum of three squares. *(2 marks)*

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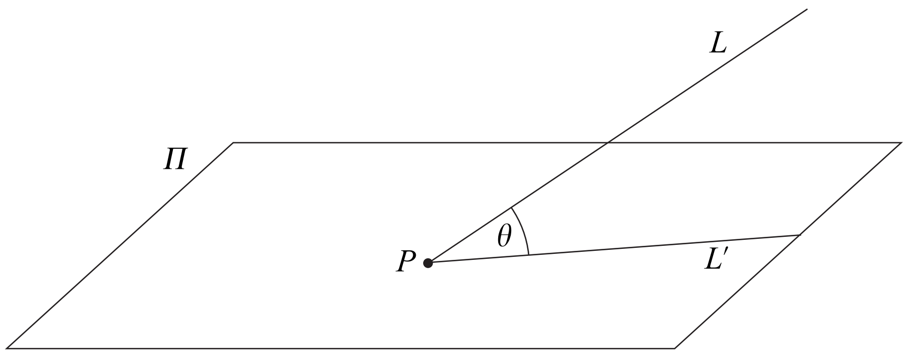
8 The diagram shows the plane Π and the lines L and L'. The plane Π and the line L have equations

$$\mathbf{r} \cdot \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix} = 37 \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

The line L does not lie in Π, and intersects it at the point P.



(a) Determine the value of θ, the angle between L and Π, giving your answer to the nearest 0.1°. *(4 marks)*

(b) Find the coordinates of P. *(4 marks)*

(c) The line L' lies in Π and is such that the angle between L and L' is θ, the angle between L and Π.

(i) Find a vector which is parallel to Π and perpendicular to L. *(3 marks)*

(ii) Hence, or otherwise, find a vector equation for L' in the form $\mathbf{r} = \mathbf{a} + \mu\mathbf{b}$. *(4 marks)*

END OF QUESTIONS

Key to mark scheme abbreviations

M	mark is for method
m or dM	mark is dependent on one or more M marks and is for method
A	mark is dependent on M or m marks and is for accuracy
B	mark is independent of M or m marks and is for method and accuracy
E	mark is for explanation
√or ft or F	follow through from previous incorrect result
CAO	correct answer only
CSO	correct solution only
AWFW	anything which falls within
AWRT	anything which rounds to
ACF	any correct form
AG	answer given
SC	special case
OE	or equivalent
A2,1	2 or 1 (or 0) accuracy marks
–*x* EE	deduct *x* marks for each error
NMS	no method shown
PI	possibly implied
SCA	substantially correct approach
c	candidate
sf	significant figure(s)
dp	decimal place(s)

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

Q	Solution	Marks	Total	Comments
1(a)	det **A** = 5p – 1 det **B** = p^2 – 10p – 11	B1 M1A1	 **3**	 M1A0 if num error(s) made
(b)	Use of det(**AB**) = det **A** det **B** Finding three values of p $p = \frac{1}{5}$, 11, –1	B1 M1 A1F	 **3**	PI Allow correct factors here ft numerical errors in (a)
	Total		**6**	
2	$\begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$ & $\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$	B1		used or written down
	Mult'n of these in the correct order	B1		at least two entries correct
	Use of addition formulae	M1		At least once
	$\begin{bmatrix} \cos(2\alpha+\beta) & \sin(2\alpha+\beta) \\ \sin(2\alpha+\beta) & -\cos(2\alpha+\beta) \end{bmatrix}$	A1F		ft only for use of clockwise rot'n and/or mult'n in wrong order
	Reflection ... ... in $y = x\tan\left(\alpha + \frac{1}{2}\beta\right)$	A1F A1F	 **6**	ft as above ft as above
	Total		**6**	
3(a)	Vector product attempted	M1		
	$\mathbf{p} \times \mathbf{q} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \times \begin{bmatrix} 7 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 30 \\ 45 \\ -30 \end{bmatrix}$	A1		
	$\ldots = 15\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$, so t = –2	A1	**3**	
(b)	Scalar triple product attempted	M1		OE, eg determinant
	$\mathbf{p} \times \mathbf{q} \bullet \mathbf{r} = 15\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 3 \\ t \end{bmatrix} = 15(13 - 2t)$	A1		
	... = 0, so $t = 6\frac{1}{2}$	A1	**3**	
	ALT: 5**p** + **q** = 6**r** ... ⇒ $t = 6\frac{1}{2}$	B2,0 B1		or any correct linear relationship
	Total		**6**	

Q	Solution	Marks	Total	Comments
4(a)	$\begin{vmatrix} 2 & 1 & 3 \\ 5 & -2 & a+1 \\ a & 2 & 4 \end{vmatrix} = a^2 + 3a - 10$	M1 A1		Attempt at det of coeff matrix Correct (accept unsimplified)
	Equating to 0 and solving quadratic in a $a = 2, -5$	m1 A1	**4**	**SC: B1** for verifying $a = 2$ **B1** for verifying $a = -5$
(b)	$\begin{aligned} 2x + y + 3z &= 3 \\ 5x - 2y + 3z &= 3 \\ 2x + 2y + 4z &= b \end{aligned}$	B1		
	Eliminations leading to two equations in two variables	M1		
	Further elimination leading to value of b	m1		
	$b = 4$	A1	**4**	
	ALT: Finding two variables in terms of third	M1		eg $y = x$ and $z = 1 - x$
	Substituting into third equation	m1		
	$b = 4$	A1		
			8	
5(a)	**(i)** Characteristic eqn $\lambda^2 - 9\lambda + 14 = 0$ $\lambda = 2, 7$	M1A1 A1		M1A0 if num error(s) made
	Substituting back for at least one eval	m1		for $\lambda = 2$, $-x + 3y = 0$ or for $\lambda = 7$, $-2x + y = 0$
	evecs $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	A1A1	**6**	or non-zero multiples
	(ii) $\mathbf{U} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$	B1FB1F		Columns of **U** and **D** are interchangeable, but must match; ft wrong answers in (i)
	$\mathbf{U}^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$	B1F B1F	**4**	1/det **U** adjoint matrix; ft incorrect **U** (provided det ≠ 0)
(b)	**(i)** evals of $\mathbf{M}^3$ are λ^3, μ^3	B1		
	since $\mathbf{M}^3 = \mathbf{U}\,\mathbf{D}^3\,\mathbf{U}^{-1}$	E1	**2**	
	(ii) evecs of $\mathbf{M}^3$ are $\mathbf{v}_1$ and $\mathbf{v}_2$	B1	**1**	
			13	

Q	Solution	Marks	Total	Comments
6 (a)(i)	$\mathbf{r} = \begin{bmatrix}1\\2\\3\end{bmatrix} + \lambda\begin{bmatrix}2\\3\\6\end{bmatrix}$	B2,1	**2**	Any correct vector line equation; B1 if one vector correct, or if both correct but equation not in correct form
	(ii) $\mathbf{r} = \begin{bmatrix}1 & 4 & -3\\2 & -1 & 0\\1 & 1 & -1\end{bmatrix}\begin{bmatrix}1+2\lambda\\2+3\lambda\\3+6\lambda\end{bmatrix} = \begin{bmatrix}-4\lambda\\\lambda\\-\lambda\end{bmatrix}$	M1 A1 A1		Attempt at multiplication At least one entry correct All three correct
	Clear and valid explanation that this is a line through O	E1	**4**	
(b) (i)	$\begin{bmatrix}1 & 4\\2 & -1\end{bmatrix}\begin{bmatrix}p\\\frac{1}{2}p+k\end{bmatrix} = \begin{bmatrix}3p+4k\\\frac{3}{2}p-k\end{bmatrix}$	B1 M1A1		For LHS For RHS
	Answer satisfies $y = \frac{1}{2}x - 3k$	A1	**4**	
	(ii) Equal gradients, hence parallel	E1F		ft if previous answer is of the form $y = \frac{1}{2}x + c$
	Distance = $\lvert k - c\rvert \cos\theta$ with $\tan\theta = \frac{1}{2}$	M1		Allow incorrect value of c here
	$\ldots = \dfrac{8k}{\sqrt{5}}$	A1	**3**	Allow $3.58k$
			13	
7 (a)(i)	Appropriate row/column operation	M1		eg $R_1{}' = R_1 + R_3$, $R_3{}' = R_3 + R_1$ or $C_3{}' = C_3 - nC_2$
	$\Delta = \begin{vmatrix}n^2+n+1 & 0 & 0\\0 & 1 & n\\1 & -(n+1) & 1\end{vmatrix}$			
	$\ldots = (n^2+n+1)\begin{vmatrix}1 & 0 & 0\\0 & 1 & n\\1 & -(n+1) & 1\end{vmatrix}$	A1	**2**	Factor correctly extracted
(ii)	Expanding remaining determinant	M1		OE
	$\Delta = (n^2 + n + 1)^2$	A1	**2**	
(b)	$\Delta = (n^2 + n)^2 + 2n^2 + 2n + 1$ $\ldots = (n^2 + n)^2 + (n + 1)^2 + n^2$	B1 B1	**2**	Accept unsimplified
(c)	Setting $n = 10$	M1		
	$111^2 = 12321 = 110^2 + 11^2 + 10^2$	A1	**2**	
			8	

Q	Solution	Marks	Total	Comments
8(a)	Use of sin or $\cos\theta = \dfrac{\text{scalar product}}{\text{product of moduli}}$	M1		using $\begin{bmatrix}3\\-2\\6\end{bmatrix}$ and $\begin{bmatrix}2\\1\\2\end{bmatrix}$
	Numerator = 16, denominator = 21	B1B1		Allow numerator $\sqrt{185}$
	$\sin\theta = \dfrac{16}{21} \Rightarrow \theta \approx 49.6^{o}$	A1	**4**	Allow AWRT 49.6
(b)	$\begin{bmatrix}2\lambda+1\\\lambda+2\\2\lambda-7\end{bmatrix} \bullet \begin{bmatrix}3\\-2\\6\end{bmatrix} = 37$	M1		
	$6\lambda + 3 - 2\lambda - 4 + 12\lambda - 42 = 37$	m1		with attempt to solve
	$\ldots \Rightarrow \lambda = 5$	A1		
	giving $P = (11, 7, 3)$	B1F	**4**	ft wrong value of λ
(c)(i)	Use of the vectors $\begin{bmatrix}3\\-2\\6\end{bmatrix}$ and $\begin{bmatrix}2\\1\\2\end{bmatrix}$	M1		
	Vector product attempted	m1		OE
	Required vector is $\begin{bmatrix}-10\\6\\7\end{bmatrix}$	A1	**3**	Or a non-zero multiple
(ii)	$\mathbf{a} = \begin{bmatrix}11\\7\\3\end{bmatrix}$	B1F		ft wrong answer in (b)
	$\mathbf{b} = \begin{bmatrix}-10\\6\\7\end{bmatrix} \times \begin{bmatrix}3\\-2\\6\end{bmatrix} = \begin{bmatrix}50\\81\\2\end{bmatrix}$	M1A1F		Or a non-zero multiple; ft wrong answer to (c)(i)
	Fully correct equation for L'	A1	**4**	
			15	
	TOTAL		**75**	

AQA – Further pure 4 – Jun 2011 – Answers

<table>
<tr><th>Question 1:</th><th>Exam report</th></tr>
<tr><td>

$a)\ \det(\mathbf{A}) = \begin{vmatrix} 1 & p & 4 \\ -3 & 2 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 1(2+1) - p(-3-2) + 4(3-4)$

$\det(\mathbf{A}) = 3 + 3p + 2p - 4 = 5p - 1$

$\det(\mathbf{B}) = \begin{vmatrix} p & 1 & 5 \\ 9 & p & -1 \\ 2 & 0 & 1 \end{vmatrix} = p(p+0) - (9+2) + 5(0-2p)$

$\det(\mathbf{B}) = p^2 - 11 - 10p = p^2 - 10p - 11$

$b)$ **AB** is singular when $\det(\mathbf{AB}) = 0$

$\det(\mathbf{AB}) = \det(\mathbf{A}) \times \det(\mathbf{B}) = (5p-1)(p^2 - 10p - 11)$

$\det(\mathbf{AB}) = (5p-1)(p-11)(p+1) = 0$

$p = \frac{1}{5}$ or $p = 11$ or $p = -1$

</td><td>Most candidates scored highly on this question. In part (b), almost all candidates used the product of the two determinants found in part (a), but a substantial number showed a lack of maturity by expanding the product into a cubic expression and then attempting to factorise it, not always correctly.</td></tr>
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<tr><th>Question 2:</th><th>Exam report</th></tr>
<tr><td>

The reflection in the line $y = x\,Tan\alpha$ has matrix

$\mathbf{A} = \begin{pmatrix} Cos2\alpha & Sin2\alpha \\ Sin2\alpha & -Cos2\alpha \end{pmatrix}$ (*formulae booklet*)

The rotation about O through an angle β has matrix

$\mathbf{B} = \begin{pmatrix} Cos\beta & -Sin\beta \\ Sin\beta & Cos\beta \end{pmatrix}$

The matrix which represents the transformation T

is $\mathbf{T} = \mathbf{BA} = \begin{pmatrix} Cos\beta & -Sin\beta \\ Sin\beta & Cos\beta \end{pmatrix} \begin{pmatrix} Cos2\alpha & Sin2\alpha \\ Sin2\alpha & -Cos2\alpha \end{pmatrix}$

$\mathbf{T} = \begin{pmatrix} Cos2\alpha Cos\beta - Sin2\alpha Sin\beta & Sin2\alpha Cos\beta + Sin\beta Cos2\alpha \\ Sin\beta Cos2\alpha + Cos\beta Sin2\alpha & Sin\beta Sin2\alpha - Cos\beta Cos2\alpha \end{pmatrix}$

$\mathbf{T} = \begin{pmatrix} Cos(2\alpha+\beta) & Sin(2\alpha+\beta) \\ Sin(2\alpha+\beta) & -Cos(2\alpha+\beta) \end{pmatrix}$

The transformation T is a reflection in the line wiht equation

$y = x\,Tan(\alpha + \frac{\beta}{2})$

</td><td>This question was not answered at all well by many candidates. Some failed to write down the matrices correctly and some failed to carry out the required matrix multiplication with the matrices in the right order. Then, relatively few candidates spotted the need to use the addition formulae, though those who did were mostly successful in giving the correct geometrical interpretation of the resulting matrix.</td></tr>
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Question 3:	Exam report
$a)\ \mathbf{p}\times\mathbf{q}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 4 & 7\\ 7 & -2 & 4\end{vmatrix}=30\mathbf{i}+45\mathbf{j}-30\mathbf{k}$ $\mathbf{p}\times\mathbf{q}=15\begin{pmatrix}2\\3\\-2\end{pmatrix}=15\mathbf{r}\ \text{when}\ t=-2$ $b)$ $\mathbf{p},\mathbf{q}$ *and* $\mathbf{r}$ are linearly dependent when $\mathbf{r}.(\mathbf{p}\times\mathbf{q})=0$. $\mathbf{r}.(\mathbf{p}\times\mathbf{q})=\begin{pmatrix}2\\3\\t\end{pmatrix}.\begin{pmatrix}30\\45\\-30\end{pmatrix}=60+135-30t=195-30t$ $\mathbf{r}.(\mathbf{p}\times\mathbf{q})=0 \Leftrightarrow 195-30t=0 \Leftrightarrow t=\frac{195}{30}=\frac{13}{2}$	Part (a) of this question was very well answered, though some candidates used a second vector product to establish the parallelism rather than simply taking out a factor 15 from their first vector product. In part (b), many candidates used the vector product already found and formed a scalar triple product which led quickly to the required value of *t*, but others seemed to be more familiar with the use of a determinant for this type of question, and this approach was equally successful though possibly taking up slightly more time.

Question 4:	Exam report
$a)$ The system does not have a unique solution when the determinant of the associated matrix $=0$ $\det=\begin{vmatrix}2 & 1 & 3\\ 5 & -2 & a+1\\ a & 2 & 4\end{vmatrix}=\begin{vmatrix}2 & 1 & 3\\ 5 & -2 & a+1\\ 5+a & 0 & 5+a\end{vmatrix}=(5+a)\begin{vmatrix}2 & 1 & 3\\ 5 & -2 & a+1\\ 1 & 0 & 1\end{vmatrix}$ $Combination: R_3^{'}=R_2+R_3\ \text{then}\ C_3^{'}=C_3-C_1$ $\det=(5+a)\begin{vmatrix}2 & 1 & 1\\ 5 & -2 & a-4\\ \boxed{1} & 0 & 0\end{vmatrix}=(5+a)\times(a-4+2)=(5+a)(a-2)$ $\det=0 \Leftrightarrow a=-5\ \text{or}\ a=2$ $b)$ When $a=2$, the system becomes $\begin{cases}2x+y+3z=3 & l_1\\ 5x-2y+3z=3 & l_2\\ 2x+2y+4z=b & l_3\end{cases} \Leftrightarrow \begin{cases}2x+y+3z=3 & \\ 9x+9z=9 & (2l_1+l_2)\\ 2x+2z=6-b & (2l_1-l_2)\end{cases} \Leftrightarrow \begin{cases}2x+y+3z=3\\ x+z=1\\ x+z=\frac{6-b}{2}\end{cases}$ The system is consistent (has infinitely many solutions) means that $\frac{6-b}{2}=1$ *this gives* $b=4$	In part (a), most candidates realised that the determinant of the coefficients on the left-hand sides of the equations must be equated to zero, and thus obtained the required values efficiently. Solutions to part (b) were often marred by a failure to indicate which equations were being used at each step. Some candidates used the slightly dubious technique of replacing one of the variables by zero, while others produced a whole page of work and still failed to reach a value for *b*.

Question 5:	Exam report
$a)i)$ Let's solve $\det(\mathbf{A}-\lambda\mathbf{I})=0$ $\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}1-\lambda & 3\\ -2 & 8-\lambda\end{vmatrix}=0$ $\Leftrightarrow (1-\lambda)(8-\lambda)+6=0$ $\Leftrightarrow \lambda^2-9\lambda+14=0$ $\Leftrightarrow (\lambda-7)(\lambda-2)=0$ $\Leftrightarrow \lambda=7 \; or \; \lambda=2$	
$ii)$ To find the eigenvectors we solve $(\mathbf{A}-\lambda\mathbf{I})\begin{pmatrix}x\\ y\end{pmatrix}=0$ For λ=2, $(\mathbf{A}-2\mathbf{I})\begin{pmatrix}x\\ y\end{pmatrix}=0 \Leftrightarrow \begin{pmatrix}-1 & 3\\ -2 & 6\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=0$ $\Leftrightarrow \begin{cases}-x+3y=0\\ -2x+6y=0\end{cases}$ An eigenvector is $\begin{pmatrix}3\\ 1\end{pmatrix}$ For λ=7, $(\mathbf{A}-7\mathbf{I})\begin{pmatrix}x\\ y\end{pmatrix}=0 \Leftrightarrow \begin{pmatrix}-6 & 3\\ -2 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=0$ $\Leftrightarrow \begin{cases}-6x+3y=0\\ -2x+y=0\end{cases}$ An eigenvector is $\begin{pmatrix}1\\ 2\end{pmatrix}$ Therfore, $\mathbf{D}=\begin{pmatrix}2 & 0\\ 0 & 7\end{pmatrix}$ *and* $\mathbf{U}=\begin{pmatrix}3 & 1\\ 1 & 2\end{pmatrix}$ $\det(\mathbf{U})=6-1=5$ *and* $\mathbf{U}^{-1}=\frac{1}{5}\begin{pmatrix}2 & -1\\ -1 & 3\end{pmatrix}$ $\mathbf{A}=\mathbf{UDU}^{-1}$ $b)i)$ $\mathbf{M}=\mathbf{UDU}^{-1}$ *and* $\mathbf{M}^3=\mathbf{UD}^3\mathbf{U}^{-1}$ *with* $\mathbf{D}^3=\begin{pmatrix}\lambda^3 & 0\\ 0 & \mu^3\end{pmatrix}$ The eigenvalues of $\mathbf{M}^3$ *are* λ^3 *and* μ^3. $ii)$ The eigenvectors of $\mathbf{M}^3$ *are* $\mathbf{v}_1$ *and* $\mathbf{v}_2$.	Candidates generally showed a very good grasp of eigenvalues and eigenvectors in part (a)(i), though some were confused when forming the eigenvectors. The responses to part (a)(ii) were almost equally successful, but some faltered in tackling the inverse of a 2 by 2 matrix, either failing to include the reciprocal of the determinant or incorrectly manipulating the elements of the matrix to form the adjoint matrix. In part (b)(i), many candidates earned partial credit, either by giving the eigenvalues but failing to show the connection with the diagonalised form, or by doing some good work with the diagonalised form and then failing to state clearly what the eigenvalues were. Part (b)(ii) was answered correctly in many cases but a common error was to write down the meaningless answers '$\mathbf{v}_1{}^3$' and '$\mathbf{v}_2{}^3$'. Some candidates used the matrix from part (a) in their answers to part (b), but this was accepted as they could still show whether they understood what was needed in part (b).

Question 6:

$a)$ A direction vector of this line is $\mathbf{u}=\begin{pmatrix}2\\3\\6\end{pmatrix}$ *and*

the point $(1,2,3)$ belongs to the line.

A vector equation of this line is $\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+t\begin{pmatrix}2\\3\\6\end{pmatrix}$

$ii)$ A point P belonging to the line has position vector $\begin{pmatrix}1+2t\\2+3t\\3+6t\end{pmatrix}$

The image of this point through the transformation U is

$$\begin{pmatrix}1&4&-3\\2&-1&0\\1&1&-1\end{pmatrix}\begin{pmatrix}1+2t\\2+3t\\3+6t\end{pmatrix}=\begin{pmatrix}1+2t+8+12t-9-18t\\2+4t-2-3t+0\\1+2t+2+3t-3-6t\end{pmatrix}=\begin{pmatrix}-4t\\t\\-t\end{pmatrix}$$

This is the line with equation $\mathbf{r}=\begin{pmatrix}0\\0\\0\end{pmatrix}+t\begin{pmatrix}-4\\1\\-1\end{pmatrix}$

This line goes through the origin O.

$b)$ $y=\frac{1}{2}x+k$. Consider a point belonging to this line $\mathbf{p}=\begin{pmatrix}x\\\frac{1}{2}x+k\end{pmatrix}$

Its image through V is : $\begin{pmatrix}1&4\\2&-1\end{pmatrix}\begin{pmatrix}x\\\frac{1}{2}x+k\end{pmatrix}=\begin{pmatrix}x+2x+4k\\2x-\frac{1}{2}x-k\end{pmatrix}$

$$=\begin{pmatrix}3x+4k\\\frac{3}{2}x-k\end{pmatrix}=\begin{pmatrix}x'\\y'\end{pmatrix}.$$

Now let's express y' in terms of x'

$2y'=3x-2k \quad and \quad x'=3x+4k$

$so \;\; x'-2y'=6k \quad or \quad y'=\frac{1}{2}x'-3k$

$ii)$ The two lines have the same gradient, they are parallel.

With the help of a sketch, we determine that

$d=(k+3k)Cos\theta$ where θ is the angle made by the line and the x-axis, $\text{Tan}\,\theta=\frac{1}{2}$.

$$\text{If Tan}\,\theta=\frac{1}{2},\; Cos\theta=\frac{1}{\sqrt{1+Tan^2\theta}}=\frac{1}{\sqrt{\frac{5}{4}}}=\frac{2}{\sqrt{5}}$$

Remember: $\frac{1}{Cos^2\theta}=1+Tan^2\theta$

$so\;\; d=4k\times\frac{2}{\sqrt{5}}=\frac{8k}{\sqrt{5}}$

Exam report

In part (a)(i), most candidates knew how to convert the given equations into a suitable vector equation, but some lost a mark by writing '*L* = ...' rather than '**r** = ...'. Part (a)(ii) was likewise answered well but often not perfectly, the usual fault here being a failure to show clearly why the line must pass through the origin.

Part (b) was found more difficult than part (a), even though it was in a two-dimensional context. Much confusion arose from the use of the same letters *x* and *y* to denote the coordinates of a general point before and after the transformation. Many candidates were able to carry out an appropriate matrix multiplication but were unsuccessful in their attempts to form a linear equation for the image line. In part (b)(ii), most candidates earned the first mark by comparing the gradients of the two lines, even if the equation of the second line was incorrect, but very few indeed realised that the 'distance' between the two parallel lines referred to the perpendicular distance between them.

Question 7:	Exam report
$a)i)\ \Delta = \begin{vmatrix} n(n+1) & n+1 & -1 \\ 0 & 1 & n \\ 1 & -(n+1) & 1 \end{vmatrix} = \begin{vmatrix} n^2+n+1 & 0 & 0 \\ 0 & 1 & n \\ 1 & -(n+1) & 1 \end{vmatrix}$ $R_1' = R_1 + R_3$ $so\ \Delta = (n^2+n+1)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & n \\ 1 & -(n+1) & 1 \end{vmatrix} = (n^2+n+1)(1+n(n+1))$ $\Delta = (n^2+n+1)(n^2+n+1) = (n^2+n+1)^2$ $b)\begin{vmatrix} n(n+1) & n+1 & -1 \\ 0 & 1 & n \\ 1 & -(n+1) & 1 \end{vmatrix} = n(n+1)\left(1+n(n+1)\right)+0+1\left(n(n+1)+1\right)$ $\Delta = \left(n(n+1)\right)^2 + 2n(n+1)+1$ $f(n) = 2n(n+1)+1 = 2n^2+2n+1 = (n+1)^2+n^2$ $\Delta = \left(n(n+1)\right)^2 + (n+1)^2 + n^2$ $c)12321 = 111^2 = \left(10^2+10+1\right)^2 = \Delta\ when\ n=10$ $so\ 12321 = \left(n(n+1)\right)^2 + (n+1)^2 + n^2 = 110^2 + 11^2 + 10^2$	Most candidates were successful in part (a), though some used more efficient methods than others. Many made a good start in part (b) but were unable to express $2n^2 + 2n + 1$ as the sum of two squares, thus losing the last three marks in the question. Some candidates seemed to be unaware that the 'squares' asked for here had to be the squares of polynomials, or in part (c) the squares of integers.

Question 8:

$a)$ A normal vector to Π is $\mathbf{n}=\begin{pmatrix}3\\-2\\6\end{pmatrix}$

and a direction vector to L is $\mathbf{u}=\begin{pmatrix}2\\1\\2\end{pmatrix}$

Let's work out the angle between these vectors:

$$\mathbf{n.u}=\begin{pmatrix}3\\-2\\6\end{pmatrix}.\begin{pmatrix}2\\1\\2\end{pmatrix}=6-2+12=16$$

$$|\mathbf{n}|=\sqrt{9+4+36}=7 \ \ and \ \ |\mathbf{u}|=\sqrt{4+1+4}=3$$

$$Cos\theta=\frac{\mathbf{n.u}}{|\mathbf{n}||\mathbf{u}|}=\frac{16}{7\times 3}=\frac{16}{21} \ \ this \ \ gives \ \ \theta=40.36$$

The angle between the line L and the plane Π is $90-40.36=49.6^{o}$

$b)$ A point on the line has position vector $\mathbf{p}=\begin{pmatrix}1+2\lambda\\2+\lambda\\-7+2\lambda\end{pmatrix}$

and the point belongs also to the plane when $\begin{pmatrix}1+2\lambda\\2+\lambda\\-7+2\lambda\end{pmatrix}.\begin{pmatrix}3\\-2\\6\end{pmatrix}=37$

$$\Leftrightarrow 3+6\lambda-4-2\lambda-42+12\lambda=37 \Leftrightarrow 16\lambda-43=37 \Leftrightarrow \lambda=5$$

The coordinates of P are P(11, 7, 3)

c) $i)$ A vector parallel to Π is perpendicular to $\mathbf{n}$

so the vector we are looking for is perpendicular to both $\mathbf{n}$ and $\mathbf{u}$

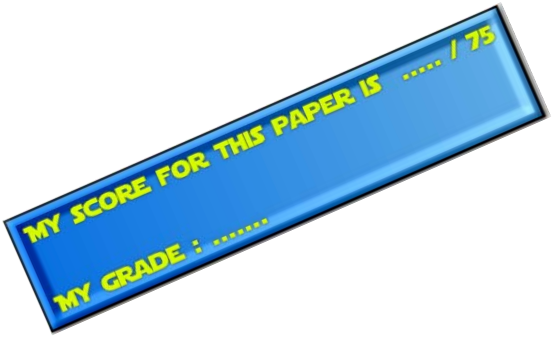
such a vector is $\mathbf{n}\times\mathbf{u}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&-2&6\\2&1&2\end{vmatrix}=-10\mathbf{i}+6\mathbf{j}+7\mathbf{k}$

$ii)$ A direction vector of the line L' is $\begin{pmatrix}-10\\6\\7\end{pmatrix}\times\begin{pmatrix}3\\-2\\6\end{pmatrix}=\begin{pmatrix}50\\81\\2\end{pmatrix}$

and an equation of this line is $\mathbf{r}=\begin{pmatrix}11\\7\\3\end{pmatrix}+t\begin{pmatrix}50\\81\\2\end{pmatrix}$

Exam report

The first two parts of this question gave almost all candidates the chance to use techniques with which they were clearly very familiar. The last mark in part (a) was often lost by a failure to subtract the inverse cosine from 90°, or equivalently to use the inverse sine.
In part (c)(i), many candidates saw that the required vector could be found by using a vector product, but others formed linear equations which they then struggled to solve. Part (c)(ii) was beyond the grasp of most candidates, though they could pick up one mark by forming an equation of a line passing through *P*. Very few candidates saw the need to form another vector product from the direction vector of the line *L* and the vector found in part (c)(i).



Grade boundaries

Grade		A*	A	B	C	D	E
Mark	Max 75	68	61	53	46	39	32