General Certificate of Education
Advanced Level Examination
June 2010

Mathematics MFP4

Unit Further Pure 4

Tuesday 15 June 2010 9.00 am to 10.30 am

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

1 The position vectors of the points P, Q and R are, respectively,

$$\mathbf{p} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

(a) Show that $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are linearly dependent. *(2 marks)*

(b) Determine the area of triangle PQR. *(4 marks)*

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2 Let $\mathbf{A} = \begin{bmatrix} 1 & x \\ 2 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 4-4x & 8 \\ 8x-4 & 4 \end{bmatrix}$.

(a) Find $\mathbf{AB}$ in terms of x. *(2 marks)*

(b) Show that $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \mathbf{C}$ for some value of x. *(5 marks)*

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3 The plane Π_1 is perpendicular to the vector $9\mathbf{i} - 8\mathbf{j} + 72\mathbf{k}$ and passes through the point $A(2, 10, 1)$.

(a) Find, in the form $\mathbf{r}\,.\,\mathbf{n} = d$, a vector equation for Π_1. *(3 marks)*

(b) Determine the exact value of the cosine of the acute angle between Π_1 and the plane Π_2 with equation $\mathbf{r}\,.\,(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 11$. *(4 marks)*

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4 The fixed points A and B and the variable point C have position vectors

$$\mathbf{a} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 2-t \\ t \\ 5 \end{bmatrix}$$

respectively, relative to the origin O, where t is a scalar parameter.

(a) Find an equation of the line AB in the form $(\mathbf{r} - \mathbf{u}) \times \mathbf{v} = \mathbf{0}$. *(3 marks)*

(b) Determine $\mathbf{b} \times \mathbf{c}$ in terms of t. *(4 marks)*

(c) (i) Show that $\mathbf{a}\,.\,(\mathbf{b} \times \mathbf{c})$ is constant for all values of t, and state the value of this constant. *(2 marks)*

(ii) Write down a geometrical conclusion that can be deduced from the answer to part **(c)(i)**. *(1 mark)*

5 Factorise fully the determinant $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}$. *(8 marks)*

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6 The line L and the plane Π have vector equations

$$\mathbf{r} = \begin{bmatrix} 7 \\ 8 \\ 50 \end{bmatrix} + t \begin{bmatrix} 6 \\ 2 \\ -9 \end{bmatrix} \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} -2 \\ 0 \\ -25 \end{bmatrix} + \lambda \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$$

respectively.

(a) (i) Find direction cosines for L. *(2 marks)*

(ii) Show that L is perpendicular to Π. *(3 marks)*

(b) For the system of equations

$$\begin{aligned} 6p + 5q + r &= 9 \\ 2p + 3q + 6r &= 8 \\ -9p + 4q + 2r &= 75 \end{aligned}$$

form a pair of equations in p and q only, and hence find the unique solution of this system of equations. *(5 marks)*

(c) It is given that L meets Π at the point P.

(i) Demonstrate how the coordinates of P may be obtained from the system of equations in part **(b)**. *(2 marks)*

(ii) Hence determine the coordinates of P. *(2 marks)*

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7 The transformation T is represented by the matrix $\mathbf{M}$ with diagonalised form

$$\mathbf{M} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}$$

where $\mathbf{U} = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 27 & 0 \\ 0 & 1 \end{bmatrix}$.

(a) (i) State the eigenvalues, and corresponding eigenvectors, of $\mathbf{M}$. *(4 marks)*

(ii) Find a cartesian equation for the line of invariant points of T. *(2 marks)*

Turn over ▶

(b) Write down $\mathbf{U}^{-1}$, and hence find the matrix $\mathbf{M}$ in the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where a, b, c and d are integers. *(5 marks)*

(c) By finding the element in the first row, first column position of $\mathbf{M}^n$, prove that

$$4 \times 3^{3n+1} + 1$$

is a multiple of 13 for all positive integers n. *(5 marks)*

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8 The matrix $\begin{bmatrix} 12 & 16 \\ -9 & 36 \end{bmatrix}$ represents the transformation which is the composition, in either order, of the two plane transformations

E: an enlargement, centre O and scale factor k $(k > 0)$

and

S: a shear parallel to the line l which passes through O

Show that $k = 24$ and find a cartesian equation for l. *(7 marks)*

END OF QUESTIONS

Key to mark scheme and abbreviations used in marking

M	mark is for method		
m or dM	mark is dependent on one or more M marks and is for method		
A	mark is dependent on M or m marks and is for accuracy		
B	mark is independent of M or m marks and is for method and accuracy		
E	mark is for explanation		
√ or ft or F	follow through from previous incorrect result	MC	mis-copy
CAO	correct answer only	MR	mis-read
CSO	correct solution only	RA	required accuracy
AWFW	anything which falls within	FW	further work
AWRT	anything which rounds to	ISW	ignore subsequent work
ACF	any correct form	FIW	from incorrect work
AG	answer given	BOD	given benefit of doubt
SC	special case	WR	work replaced by candidate
OE	or equivalent	FB	formulae book
A2,1	2 or 1 (or 0) accuracy marks	NOS	not on scheme
$-x$ EE	deduct x marks for each error	G	graph
NMS	no method shown	c	candidate
PI	possibly implied	sf	significant figure(s)
SCA	substantially correct approach	dp	decimal place(s)

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded. However, there are situations in some units where part marks would be appropriate, particularly when similar techniques are involved. Your Principal Examiner will alert you to these and details will be provided on the mark scheme.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

MFP4

Q	Solution	Marks	Total	Comments
1(a)	$\begin{vmatrix} 3 & 4 & -1 \\ -1 & 2 & 2 \\ 1 & 4 & 1 \end{vmatrix} = 6 + 8 + 4 + 2 - 24 + 4$	M1		Good attempt at det M0 for | .. | = 0 and no working
	or $3(2-8) - 4(-1-2) - 1(-4-2)$ etc or $3(2-8) + 1(4+4) + 1(8+2)$ etc Correctly shown = 0	A1		
	Or $3\mathbf{p} + 4\mathbf{q} = 5\mathbf{r}$	(M1) (A1)	2	
(b)	For attempt at 2 of $(\pm)\overrightarrow{PQ}$, $\overrightarrow{PR}$, $\overrightarrow{QR}$	M1		
	Area $\Delta PQR = \frac{1}{2}\left\|\overrightarrow{QP} \times \overrightarrow{QR}\right\|$ e.g. $= \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & -3 \\ 2 & 0 & -2 \end{vmatrix} = \frac{1}{2}\left\|\pm(4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})\right\|$	M1		Formula used with attempt at a vector product of any 2 of the above (ignore missing $\frac{1}{2}$ for now)
	$= \frac{1}{2}\sqrt{4^2 + 2^2 + 4^2}$	M1		Method for finding magnitude of their relevant vector
	$= 3$	A1	4	CSO
	Total		**6**	
2(a)	$\mathbf{AB} = \begin{bmatrix} 2x+1 & 2x-1 \\ 8 & 4 \end{bmatrix}$	M1 A1	2	Good attempt (at least one entry in R_1✓) All four correct
(b)	$\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = (\mathbf{AB})^{\mathrm{T}}$ **Or** $\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ x & 3 \end{bmatrix}$	M1		
	$= \begin{bmatrix} 2x+1 & 8 \\ 2x-1 & 4 \end{bmatrix}$	A1√		Ft their (a) **Or** CAO
	$2x + 1 = 4 - 4x$ **Or** $2x - 1 = 8x - 4$	M1		Ft previous answers
	$x = \frac{1}{2}$	A1		CAO
	Checking/noting $x = \frac{1}{2}$ in other eqn.	B1	5	*Visibly*
	Total		**7**	
3(a)	Clearly identifying $\mathbf{n} = \begin{bmatrix} 9 \\ -8 \\ 72 \end{bmatrix}$	B1		
	$d = \begin{bmatrix} 9 \\ -8 \\ 72 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 10 \\ 1 \end{bmatrix} = 10$	M1 A1	3	
(b)	Use of $\dfrac{\text{Sc.prod. of normals}}{\text{prod. of their moduli}}$	M1		Must be $(9\mathbf{i} - 8\mathbf{j} + 72\mathbf{k})$, $(\mathbf{i} + \mathbf{j} + \mathbf{k})$ or their $\mathbf{n}$ from (a)
	$\text{N}^{\text{r}}. = 73$ $\text{D}^{\text{r}}. = 73\sqrt{3}$ or $\sqrt{15987}$	B1√ B1√		Ft their **n** from (a) only
	$\cos\theta = \dfrac{1}{\sqrt{3}}$	A1	4	CAO Allow unsimplified exact forms
	Total		**7**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
4(a)	(**v** =) $\pm(\mathbf{a}-\mathbf{b}) = \pm\begin{bmatrix}1\\-5\\4\end{bmatrix}$	M1 A1		M1 A0 if $\pm\overrightarrow{AB}$ found but not stated/shown this is **v**
	$\mathbf{u} = \begin{bmatrix}3\\-4\\1\end{bmatrix}$ or $\begin{bmatrix}2\\1\\-3\end{bmatrix}$	B1	3	
(b)	$\mathbf{b}\times\mathbf{c} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & -3\\ 2-t & t & 5\end{vmatrix} = \begin{bmatrix}3t+5\\3t-16\\3t-2\end{bmatrix}$	M1 A3,2,1	4	
(c) (i)	$\mathbf{a}\bullet\mathbf{b}\times\mathbf{c} = \begin{bmatrix}3\\-4\\1\end{bmatrix}\bullet\begin{bmatrix}3t+5\\3t-16\\3t-2\end{bmatrix} = 77$	M1		Or starting again: $\begin{vmatrix}3 & -4 & 1\\ 2 & 1 & -3\\ 2-t & t & 5\end{vmatrix}$
		A1	2	CAO
(ii)	*C* never lies in plane of *O*, *A*, *B* (or is a fixed distance from it) **or** Vol. //ppd. *OABC* always = 77 **or** Vol. tetrhdrn. *OABC* always = $\frac{77}{6}$ **or** *O* is never in plane of *A*, *B*, *C* **or** $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ never co-planar	B1	1	Any suitable geometrical comment Vectors ✓; points ✗
	Total		**10**	
5	$\Delta = \begin{vmatrix}x & y & z\\ x^2 & y^2 & z^2\\ yz & zx & xy\end{vmatrix}$			
	$= \begin{vmatrix}x & y-x & z-x\\ x^2 & y^2-x^2 & z^2-x^2\\ yz & z(x-y) & y(x-z)\end{vmatrix}$	M1 M1		By $C_2' = C_2 - C_1$ (eg) $C_3' = C_3 - C_1$ (eg)
	$= (y-x)(z-x)\begin{vmatrix}x & 1 & 1\\ x^2 & y+x & z+x\\ yz & -z & -y\end{vmatrix}$	A1 A1		First two factors extracted (what's left has to be correct also)
	$= (y-x)(z-x)\begin{vmatrix}x & 1 & 0\\ x^2 & y+x & z-y\\ yz & -z & z-y\end{vmatrix}$	M1		By $C_3' = C_3 - C_2$ (e.g.)
	$= (y-x)(z-x)(z-y)\begin{vmatrix}x & 1 & 0\\ x^2 & y+x & 1\\ yz & -z & 1\end{vmatrix}$	A1		3rd factor extracted
	$= (x-y)(y-z)(z-x)(xy+yz+zx)$	M1		Further *R*/*C* ops or expansion of remaining det (almost a dM1)
	Alternatives using *Cyclic Symmetry* and the *Factor Theorem* are fine	A1	8	CAO up to equivalents due to re-positioning of the signs
	Total		**8**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
6(a)(i)	$\bullet = \sqrt{6^2+2^2+9^2}$ attempted *and* $\pm\left(\frac{6}{\bullet}, \frac{2}{\bullet}, \frac{-9}{\bullet}\right)$	M1		
	$\bullet$ = 11 and all correct	A1	2	$\pm$ (0.545, 0.182, – 0.818) ok
(ii)	**Either** $\begin{bmatrix}5\\3\\4\end{bmatrix}\times\begin{bmatrix}1\\6\\2\end{bmatrix}=-3\begin{bmatrix}6\\2\\-9\end{bmatrix}$	M1 A1		Correct vector product only here
	Explaining that d.v. of L is in dirn. of Π's nml. $\Rightarrow$ $L \perp^r \Pi$	B1		
	Or $\begin{bmatrix}6\\2\\-9\end{bmatrix}\bullet\begin{bmatrix}5\\3\\4\end{bmatrix}=0$ and $\begin{bmatrix}6\\2\\-9\end{bmatrix}\bullet\begin{bmatrix}1\\6\\2\end{bmatrix}=0$	(M1) (A1)		
	Explaining that d.v. of L is $\perp^r$ to 2 (non-//) vectors in Π $\Rightarrow$ $L \perp^r \Pi$	(B1)	3	Not just stating
(b)	E.g. 6×① – ②: 46 = 34p + 27q	M1		Eliminating r from any pair of eqns.
	2×① – ③: –57 = 21p + 6q ② – 3×③: –217 = 29p – 9q	A1 A1		Any 2 correct eqns (1 mark each)
	2×④ + 9×⑤: 605 = – 121p	M1		Solving a 2×2 system (any means) in order to get values for p, q, r
	$p = -5,\ q = 8,\ r = -1$	A1	5	All 3 ✓ CAO
(c)	$7+6t = -2+5\lambda+\mu$ **(i)** $8+2t = 0+3\lambda+6\mu$ $50-9t = -25+4\lambda+2\mu$			
	$9 = -6t+5\lambda+\mu$ → $8 = -2t+3\lambda+6\mu$ $75 = 9t+4\lambda+2\mu$	M1		Including re-arrangement attempt
	i.e. the above system with $p = -t,\ q = \lambda$ and $r = \mu$	A1	2	
	(ii) Subst^g. $t = 5$ into L's eqn.			
	Or $\lambda = 8$ ***and*** $\mu = -1$ into Π's eqn.	M1		
	$P = (37, 18, 5)$	A1	2	CAO
	Total		**14**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
7(a)(i)	Evals $\lambda = 27 \quad 1$	B1		Both
	Evecs $(\alpha)\begin{bmatrix}4\\1\end{bmatrix}$ and $(\beta)\begin{bmatrix}-1\\3\end{bmatrix}$	B1 B1		
		B1	4	Correctly matched up with evals (look out for λ_1, $\mathbf{v}_1$ notations)
(ii)	$y = -3x$	B1√		Ft $4y = x$ if evecs mis-matched
	from $\lambda = 1$	B1	2	**Must** say why they have chosen this one
(b)	$\mathbf{U}^{-1} = \frac{1}{13}\begin{bmatrix}3 & 1\\-1 & 4\end{bmatrix}$	B1 B1		Det; mtx
	$\mathbf{M} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}$	M1		Including attempt to multiply (at least **U D …**)
	$= \frac{1}{13}\begin{bmatrix}4 & -1\\1 & 3\end{bmatrix}\begin{bmatrix}27 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}3 & 1\\-1 & 4\end{bmatrix}$			
	$= \frac{1}{13}\begin{bmatrix}4 & -1\\1 & 3\end{bmatrix}\begin{bmatrix}81 & 27\\-1 & 4\end{bmatrix}$ **or** $\frac{1}{13}\begin{bmatrix}108 & -1\\27 & 3\end{bmatrix}\begin{bmatrix}3 & 1\\-1 & 4\end{bmatrix}$	A1		Ft incorrect/missing $\mathbf{U}^{-1}$ for one product; ignore missing $\frac{1}{13}$ until the end
	$= \begin{bmatrix}25 & 8\\6 & 3\end{bmatrix}$	A1	5	CAO
(c)	$\mathbf{M}^n = \mathbf{U}\,\mathbf{D}^n\,\mathbf{U}^{-1}$	M1		Including attempt to multiply
	$\mathbf{D}^n = \begin{bmatrix}27^n & 0\\0 & 1\end{bmatrix}$	B1		
	$\mathbf{M}^n\,(1,1) = \frac{1}{13}\left(12\times 27^n + 1\right)$	A1		
	So $4\times 3\times 3^{3n} + 1 = 4\times 3^{3n+1} + 1$ div. by 13	E1		Legitimately so from their working, from fact that the element is an integer
	Since **M** has all integer elements, each element of $\mathbf{M}^n$ is an integer also	E1	5	Explaining *why* it must be an integer
	Total		**16**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
8	det **W** = 12.36 + 9.16 = 576 = k^2	M1		Attempt at det. = k^2
	$\Rightarrow\ k = 24$	A1	2	
	$\frac{1}{24}\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{2}{3} \\ -\frac{3}{8} & \frac{3}{2} \end{bmatrix}$	B1		
	$\begin{bmatrix} \frac{1}{2} & \frac{2}{3} \\ -\frac{3}{8} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x + \frac{2}{3}y \\ \frac{3}{2}y - \frac{3}{8}x \end{bmatrix}$	M1 A1		
	Equating this to $\begin{bmatrix} x \\ y \end{bmatrix} =$	M1		
	$y = \frac{3}{4}x$	A1		CAO
	ALT. 1			
	$\frac{1}{24}\mathbf{W} = \begin{bmatrix} \frac{1}{2} & \frac{2}{3} \\ -\frac{3}{8} & \frac{3}{2} \end{bmatrix}$	(B1)		
	$\begin{bmatrix} \frac{1}{2} & \frac{2}{3} \\ -\frac{3}{8} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} x \\ mx \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{2}{3}m\right)x \\ \left(\frac{3}{2}m - \frac{3}{8}\right)x \end{bmatrix}$	(M1) (A1)		
	Setting $y' = mx'$ and solving for m	(M1)		Get $(4m - 3)^2 = 0$
	$y = \frac{3}{4}x$	(A1)		CAO
	ALT. 2			
	$\lambda^2 - 2\lambda + 1 = 0$	(M1)		This may simply be stated or assumed
	$\Rightarrow\ \lambda = 1$ (twice)	(A1)		
	$\lambda = 1 \Rightarrow \begin{array}{l} -\frac{1}{2}x + \frac{2}{3}y = 0 \\ -\frac{3}{8}x + \frac{1}{2}y = 0 \end{array}$	(M1) (A1)		
	$y = \frac{3}{4}x$	(A1)	5	
	Total		**7**	
	TOTAL		**75**	

AQA – Further pure 4 – Jun 2010 – Answers

Question 1:	Exam report
a) The vectors $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ are linearly dependent if and only if $\mathbf{p}.(\mathbf{q}\times\mathbf{r}) = 0$. $\mathbf{p}.(\mathbf{q}\times\mathbf{r}) = \begin{vmatrix} 3 & 4 & -1 \\ -1 & 2 & 2 \\ 1 & 4 & 1 \end{vmatrix} = 3\begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} - 4\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix}$ $= 3\times -6 - 4\times -3 - 1\times -6 = -6 + 12 - 6 = 0.$ $\mathbf{p}, \mathbf{q}$ *and* $\mathbf{r}$ are linearly dependent. b) $Area\ of\ PQR = \frac{1}{2}\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|$ $\overrightarrow{PQ} = -4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ *and* $\overrightarrow{PR} = -2\mathbf{i} + 2\mathbf{k}$ $\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 2 & 3 \\ -2 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -4 & 3 \\ -2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -4 & 2 \\ -2 & 0 \end{vmatrix}\mathbf{k} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ Area of PQR $= \frac{1}{2}\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\| = \frac{1}{2}\sqrt{4^2 + (-2)^2 + 4^2} = \frac{1}{2}\sqrt{36} = 3$	This was generally done very well. The only common mistake arose when candidates quoted the area formula for a triangle as $\frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|$ but then thought that **a** and **b** were **p** and **q** (for instance) rather than $\overrightarrow{PQ}$ *and* $\overrightarrow{PR}$ (say).

Question 2:	Exam report
$a)\ \mathbf{A}\times\mathbf{B} = \begin{bmatrix} 1 & x \\ 2 & 3 \end{bmatrix}\times\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1+2x & 2x-1 \\ 8 & 4 \end{bmatrix}$ $b)\ \mathbf{B}^T\mathbf{A}^T = (\mathbf{AB})^T = \begin{bmatrix} 1+2x & 8 \\ 2x-1 & 4 \end{bmatrix} = \mathbf{C} = \begin{bmatrix} 4-4x & 8 \\ 8x-4 & 4 \end{bmatrix}$ *if and only if* $1+2x = 4-4x$ *and* $2x-1 = 8x-4$ $6x = 3$ *and* $6x = 3$ $x = \frac{1}{2}$	This was another straightforward question and was usually well-received and successfully attempted. In part (b), candidates seemed evenly split between those who recognised that $\mathbf{B}^T\mathbf{A}^T = (\mathbf{AB})^T$ and those who simply worked out $\mathbf{B}^T\mathbf{A}^T$ directly. However, the very last mark on the question was often not gained, as many failed to show *visibly* that $x = \frac{1}{2}$ worked in **both** of the equations that arose when comparing elements of the two matrices.

Question 3:	Exam report
$a)\ \mathbf{a} = \begin{pmatrix} 2 \\ 10 \\ 1 \end{pmatrix}\quad \mathbf{n} = \begin{pmatrix} 9 \\ -8 \\ 72 \end{pmatrix}\quad \mathbf{a.n} = 18 - 80 + 72 = 10$ An equation of the plan Π_1 is $\mathbf{r.n} = 10$ b) A vector perpendicular to Π_2 is $\mathbf{u} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ $Cos\theta = \frac{\mathbf{n.u}}{\|\mathbf{n}\|\|\mathbf{u}\|} = \frac{9-8+72}{\sqrt{81+64+5184}\times\sqrt{1+1+1}} = \frac{73}{73\sqrt{3}} = \frac{1}{\sqrt{3}}$ $Cos\ \theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	A similar fault was all too common here in part (a), even amongst those who presumably knew what they were doing. A lot of candidates correctly found $d = 10$ from the use of the vector 9**i** – 8**j** + 72**k**, yet still lost a mark by nowhere identifying this as the "**n**" of the question … not even to the extent of writing down the answer in the form requested. This is just carelessness. Apart from a small minority who failed to identify **n** correctly, part (b) was found to be very straightforward indeed.

Question 4:	Exam report

$$a)\ \overrightarrow{AB}=\mathbf{b}-\mathbf{a}=\begin{pmatrix}-1\\5\\-4\end{pmatrix}$$

An equation of the line AB is $\left(\mathbf{r}-\begin{pmatrix}3\\-4\\1\end{pmatrix}\right)\times\begin{pmatrix}-1\\5\\-4\end{pmatrix}=0$

$$b)\ \mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&1&-3\\2-t&t&5\end{vmatrix}=\begin{vmatrix}1&-3\\t&5\end{vmatrix}\mathbf{i}-\begin{vmatrix}2&-3\\2-t&5\end{vmatrix}\mathbf{j}+\begin{vmatrix}2&1\\2-t&t\end{vmatrix}\mathbf{k}$$

$$=(5+3t)\mathbf{i}-(16-3t)\mathbf{j}+(3t-2)\mathbf{k}$$

$$c)\ i)\ \mathbf{a}.(\mathbf{b}\times\mathbf{c})=\begin{pmatrix}3\\-4\\1\end{pmatrix}.\begin{pmatrix}5+3t\\3t-16\\3t-2\end{pmatrix}=15+9t-12t+64+3t-2=77$$

$ii)$ The volume of the parallelepiped form by OABC is constant.

(see M.S for other options)

Exam report

This proved to be a good question for all but the weakest candidates, although it is still surprising how many mistakes are made over minus signs by further mathematicians, even when armed with a calculator. A little bit of care was needed in part(b) – almost invariably with the minus sign that accompanies the **j**-component – and it was hard to recover and answer part(c)(i) sensibly if part (b)'s answer was incorrect, as the *t* failed to cancel out. Part (c)(ii), where another explanation was required, received very poor answers indeed. There are so many valid possible "geometrical" conclusions that could be offered that it was very disappointing to see so few being offered. Most candidates opted to interpret the result as one of linear independence, which isn't a geometrical statement at all. The key issue at stake here is that the answer is *always* 77, not just that it is 77 on some specific occasion, and the candidates' remarks were expected to reflect this.

Question 5:	Exam report

$$\begin{vmatrix}x&y&z\\x^2&y^2&z^2\\yz&zx&xy\end{vmatrix}=\begin{vmatrix}x&x-y&x-z\\x^2&x^2-y^2&x^2-z^2\\yz&yz-zx&yz-xy\end{vmatrix}$$

$$=(x-y)(x-z)\begin{vmatrix}x&1&1\\x^2&x+y&x+z\\yz&-z&-y\end{vmatrix}=(x-y)(x-z)\begin{vmatrix}x&1&0\\x^2&x+y&y-z\\yz&-z&y-z\end{vmatrix}$$

$$=(x-y)(x-z)(y-z)\begin{vmatrix}x&1&0\\x^2&x+y&1\\yz&-z&1\end{vmatrix}$$

$$=(x-y)(x-z)(y-z)\left(-\begin{vmatrix}x&1\\yz&-z\end{vmatrix}+\begin{vmatrix}x&1\\x^2&x+y\end{vmatrix}\right)$$

$$=(x-y)(x-z)(y-z)(xz+yz+xy)$$

Exam report

As usual, the general treatment of determinants using row/column operations was variably received: those who could, did … and generally scored at least six of the eight marks available. Those who were less clear about the topic generally scored nothing or, at best, scrabbled around for a couple of method marks somewhere. Fortunately, the proportion of candidates in the former camp continues to grow year-by-year. A major bug-bear for the examiners is the decreasing inclination among candidates to give any hint as to what they are attempting to do with the rows and/or columns, leaving it to the markers to figure it out for themselves. In simple cases such as this, it isn't too much of a problem, but in more difficult cases examiners may be unable to award any marks unless there is a **clear method**.

Question 6:	Exam report

$a)i)$ A direction vector of L is $\mathbf{u}=\begin{pmatrix}6\\2\\-9\end{pmatrix}$

$|\mathbf{u}|=\sqrt{36+4+81}=11$

The direction cosines are $\frac{6}{11},\frac{2}{11},\frac{-9}{11}$

$$ii)\ \begin{pmatrix}5\\3\\4\end{pmatrix}\times\begin{pmatrix}1\\6\\2\end{pmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\5&3&4\\1&6&2\end{vmatrix}=\begin{vmatrix}3&4\\6&2\end{vmatrix}\mathbf{i}-\begin{vmatrix}5&4\\1&2\end{vmatrix}\mathbf{j}+\begin{vmatrix}5&3\\1&6\end{vmatrix}\mathbf{k}$$

$$=-18\mathbf{i}-6\mathbf{j}+27\mathbf{k}=-3\mathbf{u}$$

L is perpendicular to the plane Π.

$$b)\begin{cases}6p+5q+r=9 & l_1\\ 2p+3q+6r=8 & l_2\\ -9p+4q+2r=75 & l_3\end{cases}$$

$$\begin{cases}6p+5q+r=9 & l_1\\ 34p+27q=46 & 6l_1-l_2\\ 21p+6q=-57 & 2l_1-l_2\end{cases}$$

This gives $\begin{cases}34p+27q=46\\7p+2q=-19\end{cases}$ $\begin{cases}68p+54q=92\\-189p-54q=513\end{cases}$

By adding the equations : $-121p=605$

$p=-5$

then $7\times-5+2q=-19$ $\quad q=8$

and $r=9-6p-5q=9+30-40$ $\quad r=-1$

$c)$ By solving the equation simultaneously, we have

$$\begin{pmatrix}7+6t\\8+2t\\50-9t\end{pmatrix}=\begin{pmatrix}-2+5\lambda+\mu\\0+3\lambda+6\mu\\-25+4\lambda+2\mu\end{pmatrix}$$

$$\begin{cases}-6t+5\lambda+\mu=9\\-2t+3\lambda+6\mu=8\\9t+4\lambda+2\mu=75\end{cases}$$

This is equivalent to the system in part b) with

$t=-p,\lambda=q$ *and* $\mu=r$

so $t=-p=5$

and $\mathbf{r}=\begin{pmatrix}7+6t\\8+2t\\50-9t\end{pmatrix}=\begin{pmatrix}37\\18\\5\end{pmatrix}$ *so* $P=(37,18,5)$

Exam report

There were several easy marks to be had in this question, and the rest was helpfully signposted. Part (a)(i) was routine, as was part (a)(ii) … or it would have been had not, again, some sort of explanation been required for full marks. Approaches were evenly split between using the scalar-product and the vector-product. In the former case, a lot of candidates seemed to think they only needed to show that the line's direction vector was perpendicular to **one** vector parallel to the plane. In the second case, many candidates were content merely to find the vector product of the two vectors "in" (parallel to) the plane; others noted that it was a multiple of the d.v. of *L* without further comment. For the 3rd mark on this part of the question, we wanted an explanation as to *why* their scalar- or vector-product work established the given outcome – it is not enough for candidates merely to repeat the result that we gave them. *Given* results always require more of candidates by way of 'dotting the i's and crossing the t's'.

The more discerning teachers and students of MFP4 will understand why part (b) was set up as it was, since this is the only way to force candidates not just to resort to calculator-output generated answers. In part (c), remarkably few got past equating the equations of *L* and Π; the majority of candidates had no real idea what to do and so there were lots of answers seen in which the point of intersection was taken to be (–5, 8, –1).

Question 7:

$a)\, i)$ The eigenvalues are 27 and 1

The eigenvectors are $\begin{pmatrix}4\\1\end{pmatrix}$ *and* $\begin{pmatrix}-1\\3\end{pmatrix}$

$ii)$ The invariant line has direction vector $\begin{pmatrix}-1\\3\end{pmatrix}$ *for* $\lambda = 1$

Cartesian equations : $\dfrac{x-0}{-1} = \dfrac{y-0}{3}$ $\quad y = -3x$

$b)\ \det(\mathbf{U}) = 12 + 1 = 13$

$$\mathbf{U}^{-1} = \frac{1}{13}\begin{bmatrix}3 & 1\\-1 & 4\end{bmatrix}$$

$$\mathbf{U}\times\mathbf{D} = \begin{bmatrix}4 & -1\\1 & 3\end{bmatrix}\times\begin{bmatrix}27 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}108 & -1\\27 & 3\end{bmatrix}$$

$$\mathbf{UDU}^{-1} = \frac{1}{13}\begin{bmatrix}108 & -1\\27 & 3\end{bmatrix}\times\begin{bmatrix}3 & 1\\-1 & 4\end{bmatrix} = \frac{1}{13}\begin{bmatrix}325 & 104\\78 & 39\end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix}25 & 8\\6 & 3\end{bmatrix}$$

$$c)\ \mathbf{M}^n = \mathbf{UD}^n\mathbf{U}^{-1} \ \ and \ \ \mathbf{D}^n = \begin{bmatrix}27^n & 0\\0 & 1\end{bmatrix}$$

$$\mathbf{M}^n = \frac{1}{13}\begin{bmatrix}4 & -1\\1 & 3\end{bmatrix}\times\begin{bmatrix}27^n & 0\\0 & 1\end{bmatrix}\begin{bmatrix}3 & 1\\-1 & 4\end{bmatrix}$$

$$\mathbf{M}^n = \frac{1}{13}\begin{bmatrix}4 & -1\\1 & 3\end{bmatrix}\times\begin{bmatrix}3\times 27^n & 27^n\\-1 & 4\end{bmatrix}$$

$$so\ \mathbf{M}^n{}_{1,1} = \frac{1}{13}\left(12\times 27^n + 1\right) = \frac{1}{13}\left(4\times 3^{3n+1} + 1\right)$$

Since all the elements of $\mathbf{M}$ are integers,

all the elements of $\mathbf{M}^n$ are also integers

including $\mathbf{M}^n{}_{1,1}$ this mean that $4\times 3^{3n+1} + 1$ is a multiple of 13.

Exam report

This question was handled very well indeed in its technical aspects – writing down eigenvalues and eigenvectors; finding a 2×2 inverse matrix; determining **M**, and the accompanying matrix multiplications. So a lot of candidates scored lots of marks on this question. However, there were very few candidates indeed who managed to score more than 12 or 13, due to the explanations that were being looked for. At its simplest level, this began in part (a)(i), where candidates were required to write down "the eigenvalues and ***corresponding*** eigenvectors …" – in other words, to say which went with which. Examiners did not accept a fortuitously ordered separate pair of each (with a 50-50 chance of a correct pairing). In part (a)(ii), we also wanted to know **why** they had chosen one of the two possible lines as a line of invariant points (as opposed to just an invariant line).
Part (c) had been designed to require thought of the candidates, so the difficulties that arose here were not unexpected. There were two marks for explaining why the given expression was divisible by 13. The first was awarded to anyone who noted that the 1-1 element of $\mathbf{M}^n$ is an integer; the second, subtler, one was for the explanation of why it had to be an integer. Disappointingly few candidates got both marks.

Question 8:

If the enlargement has a scale factor k,

the area scale factor is $k^2 = \det(\mathbf{M})$

$$k^2 = \begin{vmatrix}12 & 16\\-9 & 36\end{vmatrix} = 12\times 36 + 9\times 36 = 576 \qquad k = \sqrt{576} = 24.$$

A line going through the origin is $y = mx$

This line is invariant through the shear.

$$\frac{1}{24}\mathbf{M}\times\begin{bmatrix}x\\mx\end{bmatrix} = \begin{bmatrix}\dfrac{12x+16mx}{24}\\[2mm]\dfrac{-9x+36mx}{24}\end{bmatrix} = \begin{bmatrix}x'\\mx'\end{bmatrix}$$

this gives $m\left(\dfrac{12x+16mx}{24}\right) = \dfrac{-9x+36mx}{24}$

$$12m + 16m^2 = -9 + 36m$$

$$16m^2 - 24m - 9 = 0$$

$$(4m-3)^2 = 0 \qquad m = \frac{3}{4} \ \ and \ \ l : y = \frac{3}{4}x$$

Exam report

This question was not a success for most candidates. There were a lot of blanks drawn amongst candidates, and a lot more "solutions" which consisted of little more than "hopeful manipulation" of numbers. It appeared that a large number of candidates had simply stumbled on the determinant of 576 and then pointed out that *k* must be its square root. Many candidates took the opportunity to play around with the (given) 24 in all sorts of ways, some of which only fortuitously resembled correct working. Full or coherent explanations were in very short supply.

MY SCORE FOR THIS PAPER IS ..... / 75

MY GRADE : ....

Grade boundaries

Grade		A*	A	B	C	D	E
Mark	Max 75	67	60	52	44	36	29