General Certificate of Education
June 2009
Advanced Level Examination

MATHEMATICS **MFP4**
Unit Further Pure 4

Wednesday 17 June 2009 9.00 am to 10.30 am

For this paper you must have:
- a 12-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP4.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 Let $\mathbf{P} = \begin{bmatrix} 1 & 4 & 2 \\ -1 & 2 & 6 \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} k & 1 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$, where k is a constant.

(a) Determine the product matrix $\mathbf{PQ}$, giving its elements in terms of k where appropriate. *(3 marks)*

(b) Find the value of k for which $\mathbf{PQ}$ is singular. *(2 marks)*

2 (a) Write down the 3×3 matrices which represent the transformations A and B, where:

(i) A is a reflection in the plane $y = x$; *(2 marks)*

(ii) B is a rotation about the z-axis through the angle θ, where $\theta = \frac{\pi}{2}$. *(1 mark)*

(b) (i) Find the matrix $\mathbf{R}$ which represents the composite transformation

'A followed by B' *(3 marks)*

(ii) Describe the single transformation represented by $\mathbf{R}$. *(2 marks)*

3 The plane Π has equation $\mathbf{r} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$.

(a) Find an equation for Π in the form $\mathbf{r} \cdot \mathbf{n} = d$. *(4 marks)*

(b) Show that the line with equation $\mathbf{r} = \begin{bmatrix} 7 \\ 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 10 \\ 1 \\ 5 \end{bmatrix}$ does not intersect Π, and explain the geometrical significance of this result. *(4 marks)*

4 (a) Show that the system of equations

$$\begin{aligned} 3x - y + 3z &= 11 \\ 4x + y - 5z &= 17 \\ 5x - 4y + 14z &= 16 \end{aligned}$$

does not have a unique solution and is consistent.

(You are not required to find any solutions to this system of equations.) *(4 marks)*

(b) A transformation T of three-dimensional space maps points (x, y, z) onto image points (x', y', z') such that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x - y + 3z - 2 \\ 2x + 6y - 4z + 12 \\ 4x + 11y + 4z - 30 \end{bmatrix}$$

Find the coordinates of the invariant point of T. *(8 marks)*

5 The points A, B, C and D have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ respectively, relative to the origin O, where

$$\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} \text{ and } \mathbf{d} = \begin{bmatrix} 5 \\ 5 \\ 11 \end{bmatrix}$$

(a) Using scalar triple products:

(i) show that $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are coplanar; *(2 marks)*

(ii) find the volume of the parallelepiped defined by AB, AC and AD. *(4 marks)*

(b) (i) Find the direction ratios of the line BD. *(2 marks)*

(ii) Deduce the direction cosines of the line BD. *(2 marks)*

6 The plane transformation T is defined by

$$\mathrm{T} : \begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $\mathbf{M} = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$.

(a) Evaluate $\det \mathbf{M}$ and state the significance of this answer in relation to T. *(2 marks)*

(b) Find the single eigenvalue of **M** and a corresponding eigenvector. Describe the geometrical significance of these answers in relation to T. *(5 marks)*

(c) Show that the image of the line $y = \frac{1}{2}x + k$ under T is $y' = \frac{1}{2}x' + k$. *(3 marks)*

(d) Given that T is a shear, give a full geometrical description of this transformation. *(2 marks)*

7 The 2×2 matrix **M** has an eigenvalue 3, with corresponding eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and a second eigenvalue -3, with corresponding eigenvector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

The diagonalised form of **M** is $\mathbf{M} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}$.

(a) (i) Write down suitable matrices **D** and **U**, and find $\mathbf{U}^{-1}$. *(4 marks)*

(ii) Hence determine the matrix **M**. *(3 marks)*

(b) Given that n is a positive integer, use the result $\mathbf{M}^n = \mathbf{U}\,\mathbf{D}^n\,\mathbf{U}^{-1}$ to show that:

(i) when n is even, $\mathbf{M}^n = 3^n\,\mathbf{I}$;

(ii) when n is odd, $\mathbf{M}^n = 3^{n-1}\,\mathbf{M}$. *(6 marks)*

8 (a) Matrix $\mathbf{M} = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$. Without attempting to factorise, expand fully $\det \mathbf{M}$.

(2 marks)

(b) Matrix $\mathbf{N} = \begin{bmatrix} d & e & f \\ f & d & e \\ e & f & d \end{bmatrix}$. Find the product matrix $\mathbf{MN}$. *(3 marks)*

(c) Prove that the product

$$(a^3 + b^3 + c^3 - 3abc)(d^3 + e^3 + f^3 - 3def)$$

can be written in the form $x^3 + y^3 + z^3 - 3xyz$, stating clearly each of x, y and z in terms of a, b, c, d, e and f. *(2 marks)*

END OF QUESTIONS

Key to mark scheme and abbreviations used in marking

M	mark is for method
m or dM	mark is dependent on one or more M marks and is for method
A	mark is dependent on M or m marks and is for accuracy
B	mark is independent of M or m marks and is for method and accuracy
E	mark is for explanation

√or ft or F	follow through from previous incorrect result	MC	mis-copy
CAO	correct answer only	MR	mis-read
CSO	correct solution only	RA	required accuracy
AWFW	anything which falls within	FW	further work
AWRT	anything which rounds to	ISW	ignore subsequent work
ACF	any correct form	FIW	from incorrect work
AG	answer given	BOD	given benefit of doubt
SC	special case	WR	work replaced by candidate
OE	or equivalent	FB	formulae book
A2,1	2 or 1 (or 0) accuracy marks	NOS	not on scheme
$-x$ EE	deduct x marks for each error	G	graph
NMS	no method shown	c	candidate
PI	possibly implied	sf	significant figure(s)
SCA	substantially correct approach	dp	decimal place(s)

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded. However, there are situations in some units where part marks would be appropriate, particularly when similar techniques are involved. Your Principal Examiner will alert you to these and details will be provided on the mark scheme.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

MFP4

Q	Solution	Marks	Total	Comments
1(a)	$\begin{bmatrix} 1 & 4 & 2 \\ -1 & 2 & 6 \end{bmatrix}\begin{bmatrix} k & 1 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} k+14 & -1 \\ 22-k & 3 \end{bmatrix}$	M1 A1 A1	 3	**PQ** a 2×2 matrix At least one element in C_1 correct All correct
(b)	Det(**PQ**) = $3k + 42 + 22 - k$ $= 2k + 64 = 0$ $k = -32$	M1 A1	 2	Det of a square matrix attempted and equated to zero ft in 2×2 case only (linear eqns.)
	Total		**5**	
2(a)(i)	$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	B2	2	
(ii)	$\mathbf{B} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	B1	1	
(b)(i)	$\mathbf{R} = \mathbf{BA} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	M1 A1 A1	 3	Product correct way around Most correct; all correct ft ft
(ii)	Reflection in $x = 0$ (or y–z plane)	M1 A1	2	M for correct **R**
	Note 1: For $\mathbf{R} = \mathbf{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	(B1)		If all correct, ft their **A**, **B**
	Reflection in $y = 0$ (or x–z plane)	(M1) (A1)		Full ft, M for correct **R**
	Note 2: 90° rotation in –ve sense gives $\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	(B1)		**A** as before
	$\mathbf{R} = \mathbf{BA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Reflection in $y = 0$ (or x–z plane)	(M1) (A1) (A1) (M1) (A1)		Full ft (incl. Note 1 possibility – Reflection in $x = 0$ (or y–z plane))
	Total		**8**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
3(a)	$\mathbf{n} = (3\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \times (4\mathbf{i} - \mathbf{j} + \mathbf{k})$	M1		
	$= 3\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$	A1		cao
	$d = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) \bullet (\text{their } \mathbf{n}) = 4$	M1 A1	4	ft
(b)	$\begin{bmatrix} 7+10t \\ 1+t \\ 4+5t \end{bmatrix}$ subst[d]. into their plane eqn.	M1		(In at least the LHS of it)
	$21 + 30t + 5 + 5t - 28 - 35t = 4$	dM1		Linear "eqn." in t created (LHS)
	Since $-2 \neq 4$, no intersection	A1		Explained or stated. N.B. can ft other d's (except -2) but if **n** is wrong also the t won't vanish, so no ft then
	Line parallel to plane	B1		May be independently asserted
	OR			
	$\begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix} \bullet \begin{bmatrix} 10 \\ 1 \\ 5 \end{bmatrix} = 0$	(M1) (A1) (B1)		For showing line not in plane
	Line perp[r]. to nml. $\Rightarrow$ line // to plane	(B1)		
	OR			
	$\begin{bmatrix} 7+10t \\ 1+t \\ 4+5t \end{bmatrix}$ equated to $\begin{bmatrix} 2+3\lambda+4\mu \\ 1+\lambda-\mu \\ 4+2\lambda+\mu \end{bmatrix}$	(M1)		Incl. starting to do something
	Eliminating λ, μ to get linear eqn. in t	(dM1)		
	Since $-2 \neq 4$, no intersection	(A1)		Explained or stated
	Line parallel to plane	(B1)	4	May be independently asserted
	Total		**8**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
4(a)	$3 \times [1] - [2] \Rightarrow 5x - 4y + 14z = 16$	M2 A1		Or eliminating (say) y twice to get two lots of $7x - 2z = 28$
	Giving no unique soln. *and* consistent	E1		
	For those who just show $\Delta = 0$ to conclude that there is no unique soln.	(M1) (A1)		and save the other M1 A1 for demonstrating consistency
	OR			
	Solving e.g. in [1] & [2]: $\dfrac{x-4}{2} = \dfrac{y-1}{27} = \dfrac{z}{7} = \lambda$	(M1) (A1)		
	Subst[g]. in [3] for x, y, z in terms of λ Showing LHS = RHS = 16	(M1) (A1)		$5(2\lambda + 4) - 4(1 + 27\lambda) + 14(7\lambda)$
	OR			
	$\left(\begin{array}{ccc\|c} 3 & -1 & 3 & 11 \\ 4 & 1 & -5 & 17 \\ 5 & -4 & 14 & 16 \end{array}\right) \rightarrow \left(\begin{array}{ccc\|c} 3 & -1 & 3 & 1 \\ 1 & 2 & -8 & 6 \\ -1 & -2 & 8 & -6 \end{array}\right)$	(M1) (A1) (A1)		$R_2{}' = R_2 - R_1$ $R_3{}' = R_3 - 2R_1$
	$R_2{}' = -R_3{}' \Rightarrow$ no unique soln. and consistency	(E1)		
	OR			
	Showing $\Delta = 0 \Rightarrow$ no unique soln.	(M1) (A1)		
	Attempt at each of $\Delta_x = \begin{vmatrix} 11 & -1 & 3 \\ 17 & 1 & -5 \\ 16 & -4 & 14 \end{vmatrix}$, $\Delta_y = \begin{vmatrix} 3 & 11 & 3 \\ 4 & 17 & -5 \\ 5 & 16 & 14 \end{vmatrix}$ and $\Delta_z = \begin{vmatrix} 3 & -1 & 11 \\ 4 & 1 & 17 \\ 5 & -4 & 16 \end{vmatrix}$	(M1)		
	Each shown = 0 and this $\Rightarrow$ consistency	(A1)	4	
(b)	Setting $x' = x,\ y' = y,\ z' = z$	M1		
	$2 = -y + 3z$ $-12 = 2x + 5y - 4z$ $30 = 4x + 11y + 3z$	A1		Or equivalent
	E.g. $\left.\begin{array}{l} 2 = 3z - y \\ 54 = 11z + y \end{array}\right\}$ by (3) – 2 × (2)	M1 A1		Reducing to 2×2 system; Correctly ft their system
	$z = 4$, $y = 10$	M1 A1		Solving ; correctly
	$x = -23$	M1 A1	8	Subst[g]. back to find 3rd coord.
	OR Other methods for solving a 3×3 system will be constructed should they arise			
	Total		**12**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
5(a)(i)	$\mathbf{a}\bullet\mathbf{b}\times\mathbf{c} = \begin{vmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 1 & -1 & 5 \end{vmatrix} = 0$	M1 A1	2	Legitimately shown to be zero
(ii)	$\overrightarrow{AB} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\overrightarrow{AC} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$, $\overrightarrow{AD} = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$	M1 A1		At least two correct
	Attempt at $\overrightarrow{AB} \bullet \overrightarrow{AC} \times \overrightarrow{AD}$	M1		Any order (+/–), some Sc.Trip.Pr.
	$V = 6$	A1	4	cao and not –ve
(b)(i)	$\overrightarrow{BD} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$; i.e. 2 : 3 : 6	M1 A1	2	
(ii)	$\sqrt{2^2+3^2+6^2} = 7$	M1		
	DCs are $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$	A1	2	ft
	Total		**10**	
6(a)	Det(**M**) = 1 $\Rightarrow$ Area invariant under T	B1 B1	2	2nd B1 ft ref. "area"
(b)	Char. Eqn. $\lambda^2 - 2\lambda + 1 = 0$	M1		
	$\Rightarrow \lambda = 1$ (twice)	A1		
	Subst^g. their λ back to find an evec:	M1		
	$\alpha\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	A1		Any (non-zero) α
	(Since $\lambda = 1$) this represents a line of inv. pts.	B1	5	ft if $\lambda \neq 1$
(c)	$\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ \frac{1}{2}x+k \end{bmatrix} = \begin{bmatrix} x+4k \\ \frac{1}{2}x+3k \end{bmatrix}$	M1 A1		
	Verifying that $y' = \frac{1}{2}x' + k$	A1	3	Be convinced AG
(d)	Inv. line (or parallel to) $y = \frac{1}{2}x$	B1		
	Mapping (e.g.) (1, 0) to (−1, − 1) Give 0 + 0 if called any other kind of transformation	B1	2	Any pt. not on $y = \frac{1}{2}x$ and its image
	Total		**12**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
7(a)(i)	$\mathbf{D} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$ $\mathbf{U} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$	B1 B1		**D**, **U** (alt. choices ok)
	$\mathbf{U}^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}$	B1 B1	4	ft 1st B1 provided det ≠ 0 ft 2nd B1 in non-trivial cases
(ii)	$\mathbf{M} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}$	M1		Some attempt at mtx. multn.
	$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 12 & -3 \\ 6 & -3 \end{bmatrix}$ or $\frac{1}{2}\begin{bmatrix} 3 & -3 \\ 6 & -12 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}$	A1		First multn. correct ft
	$= \begin{bmatrix} 9 & -3 \\ 24 & -9 \end{bmatrix}$	A1	3	Ft missing $\frac{1}{2}$ only
(b)(i)	When n even, $\mathbf{D}^n = \begin{bmatrix} 3^n & 0 \\ 0 & 3^n \end{bmatrix}$	M1		Incl. use in mtx. multn. of form $\mathbf{U}\,\mathbf{D}^n\,\mathbf{U}^{-1}$
	$\mathbf{M}^n = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 4.3^n & -3^n \\ -2.3^n & 3^n \end{bmatrix}$ or $\frac{1}{2}\begin{bmatrix} 3^n & 3^n \\ 2.3^n & 4.3^n \end{bmatrix}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}$ correct	A1		Correct ft
	Showing $\mathbf{M}^n = 3^n\,\mathbf{I}$ legitimately	A1	3	
(ii)	When n odd, $\mathbf{D}^n = \begin{bmatrix} 3^n & 0 \\ 0 & -3^n \end{bmatrix}$	M1		Incl. use in mtx. multn. of form $\mathbf{U}\,\mathbf{D}^n\,\mathbf{U}^{-1}$
	$\mathbf{M}^n = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 4.3^n & -3^n \\ 2.3^n & -3^n \end{bmatrix}$ or $\frac{1}{2}\begin{bmatrix} 3^n & -3^n \\ 2.3^n & -4.3^n \end{bmatrix}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}$ correct	A1		Correct ft
	Showing $\mathbf{M}^n = 3^{n-1}\,\mathbf{M}$ legitimately	A1	3	
	Total		**13**	
8(a)	Det(**M**) = $a^3 + b^3 + c^3 - 3abc$	M1 A1	2	Good attempt; correct
(b)	$\begin{bmatrix} ad+bf+ce & ae+bd+cf & af+be+cd \\ af+be+cd & ad+bf+ce & ae+bd+cf \\ ae+bd+cf & af+be+cd & ad+bf+ce \end{bmatrix}$	M1 A1 A1	3	At least 5 correct; all 9 correct
(c)	Use of det(**MN**) = det(**M**) det(**N**) $x = ad + bf + ce$, $y = ae + bd + cf$ and $z = af + be + cd$	M1 A1	2	All correctly identified Give B1 (SC) if just this with no explanation why
	Total		**7**	
	TOTAL		**75**	

AQA – Further pure 4 – Jun 2009 – Answers

Question 1:	Exam report
a) $\begin{bmatrix}1 & 4 & 2\\ -1 & 2 & 6\end{bmatrix}\times\begin{bmatrix}k & 1\\ 2 & -1\\ 3 & 1\end{bmatrix}=\begin{bmatrix}k+8+6 & 1-4+2\\ -k+4+18 & -1-2+6\end{bmatrix}=\begin{bmatrix}k+14 & -1\\ 22-k & 3\end{bmatrix}$ b) $\mathbf{PQ}$ is singular if $\det(\mathbf{PQ})=0$ $\begin{vmatrix}k+14 & -1\\ 22-k & 3\end{vmatrix}=3(k+14)+(22-k)=2k+64=0$ $k=-32$	This was a straightforward starter to the paper, and was generally found to be so by candidates. Almost all candidates knew what to do, in principle, although there were unexpectedly large numbers of arithmetical (particularly sign) errors in part (b) when finding the required value of *k*.

Question 2:	Exam report
i) Through the reflection in the plane $y=x$, the image of $\mathbf{i},\mathbf{j},\mathbf{k}$ *are* $\mathbf{j},\mathbf{i},\mathbf{k}$ *respectively* The corresponding matrix is $\mathbf{A}=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$ ii) Through the rotation about the z-axis with angle $\frac{\pi}{2}$, the image of $\mathbf{i},\mathbf{j},\mathbf{k}$ *are* $\mathbf{j},-\mathbf{i},\mathbf{k}$ *respectively* The corresponding matrix is $\mathbf{B}=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$ b) i) "$\mathbf{A}$ followed by $\mathbf{B}$" : $\mathbf{R}=\mathbf{BA}=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\times\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ ii) Through the transformation $\mathbf{R}$, $\mathbf{j}$ and $\mathbf{k}$ are invariant and $\mathbf{i}$ is mapped onto $-\mathbf{i}$. This is the reflection in the plane $x=0$ (*or the yOz plane*)	This was another fairly basic test of the results, given in the formulae booklet, relating to the matrices of 3-d transformations, and was mostly handled very well. Apart from a small minority who made elementary mistakes with the matrices for A and B, the most common error was in taking 'A followed by B' to be represented by the product **AB** rather than **BA**. A few candidates mistakenly described the reflection as being in the *y* = *z* plane when they really meant the *y*-*z* plane. In marking part (b)(ii), the error of describing *x* = 0 as a line rather than a plane was overlooked. Follow-through marks were allowed for those who had multiplied AB rather than **BA** but in **no** other cases.

Question 3:	Exam report

a) Direction vectors of the plane are $\mathbf{u}\begin{pmatrix}3\\1\\2\end{pmatrix}$ *and* $\mathbf{v}\begin{pmatrix}4\\-1\\1\end{pmatrix}$ A vector normal to the plane is $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&1&2\\4&-1&1\end{vmatrix}=3\mathbf{i}+5\mathbf{j}-7\mathbf{k}$ A point belonging to the plane Π is (3,1,2) $d=\begin{pmatrix}2\\1\\1\end{pmatrix}.\begin{pmatrix}3\\5\\-7\end{pmatrix}=6+5-7=4$ An equation of the plane is $\mathbf{r}.\begin{pmatrix}3\\5\\-7\end{pmatrix}=4$ *b*) The line does not intersect the plane Π means that the direction vector of the line is orthogonal/perpendicular to a normal vector of the plane. $Let's\ work\ out\ \begin{pmatrix}10\\1\\5\end{pmatrix}.\begin{pmatrix}3\\5\\-7\end{pmatrix}=30+5-35=35-35=0$ The line is parallel to the plane.	This was the first question on the paper which required some explanation by candidates, to justify their working in some way or to explain their results. Although responses were a *lot* better than they generally have been in the past, there is still much scope for improvement. In particular, candidates need to realise that, when the answer is given in the question, they need to be a little more diligent in its justification. This especially applies to the better candidates, who can frequently "see" things as obvious or work them out "in their heads". Sloppy presentation of solutions can often lose these candidates marks. In part (b), the question tells them that the line and plane do not intersect. Even amongst those who substituted the given line equation into the correct plane equation, and found a contradiction of the "– 2 = 4" variety (see the mark scheme), very few offered a satisfactory explanation of the non-intersection. A small number of candidates took the slightly less obvious approach of showing that the line was perpendicular to the plane's normal; however, this did not, on its own, establish that the line didn't lie *in* the plane, and an extra step was needed in order to gain all four marks here.

Question 4:

Exam report: This proved surprisingly tough for most candidates. The work is a three-dimensional extension of the linear simultaneous equations work found at GCSE level, yet the majority of candidates found the algebra too tough for their liking. Part (a) was usually handled competently, but part (b) elicited a great range of responses, possibly because of the extra layer provided by requiring recognition that (*x*′, *y*′, *z*′) = (*x*, *y*, *z*) for invariant points. Many candidates had no idea what to do with the three expressions given in the right-hand column vector and generally, explicitly or by default, took it to be equal to (0,0,0) before attempting to solve. Even then, very few gained the correct follow-through answer.

a) The system of equations is equivalent to the matrix equation

$$\mathbf{M}\times\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}11\\17\\16\end{pmatrix}\ \text{with}\ \mathbf{M}=\begin{pmatrix}3&-1&3\\4&1&-5\\5&-4&14\end{pmatrix}$$

This system does not have a unique solution when $\det(\mathbf{M})=0$.

Let's work out $\begin{vmatrix}3&-1&3\\4&1&-5\\5&-4&14\end{vmatrix}=3\begin{vmatrix}1&-5\\-4&14\end{vmatrix}+1\begin{vmatrix}4&-5\\5&14\end{vmatrix}+3\begin{vmatrix}4&1\\5&-4\end{vmatrix}$

$$=-18+81-63=-81+81=0.$$

Is the system consistent?

$$\begin{cases}3x-y+3z=11 & l_1\\4x+y-5z=17 & l_2\\5x-4y+14z=16 & l_3\end{cases}\Leftrightarrow\begin{cases}3x-y+3z=11\\7x-2z=28 & l_1+l_2\\7x-2z=28 & 4l_1-l_3\end{cases}$$

The system is consistent.

b) A point is invariant if $x'=x, y'=y\ and\ z'=z$

This gives $\begin{cases}x=x-y+3z-2\\y=2x+6y-4z+12\\z=4x+11y+4z-30\end{cases}\Leftrightarrow\begin{cases}y-3z=-2 & l_1\\2x+5y-4z=-12 & l_2\\4x+11y+3z=30 & l_3\end{cases}$

$$\Leftrightarrow\begin{cases}y-3z=-2 & l_1'\\2x+5y-4z=-12 & l_2'\\y+11z=54 & l_3'=l_3-2l_2\end{cases}$$

Then $l_1'-l_3'$ *gives* $-14z=-56$ *and* $z=4$

$$y=3z-2=3\times4-2=10$$

and $2x=-5y+4z-12=-50+16-12=-46$ *so* $x=-23$

The invariant point is $(-23,10,4)$

Question 5:

Exam report: This was the most straightforward question on the paper after question 1 and was generally handled very successfully by candidates. A few slips arose in the evaluation of scalar and/or vector products, and some candidates found the volume of the parallelepiped defined by *O*, *A*, *B* and *C* rather than *A*, *B*, *C* and *D* in part (a)(ii). Candidates who found the volume to be negative were penalised.

a) *i*) $\overrightarrow{OA},\overrightarrow{OB},\overrightarrow{OC}$ *are coplanar when* $\mathbf{a}.(\mathbf{b}\times\mathbf{c})=0$

Let's work out $\mathbf{a}.(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}2&1&4\\3&2&5\\1&-1&5\end{vmatrix}=2(10+5)-1(15-5)+4(-3-2)$

$$=30-10-20=0$$

The three vectors are coplanar.

ii) The volume of the parallelepiped is $\left|\overrightarrow{AB}.\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)\right|$.

$$\overrightarrow{AB}=\mathbf{b}-\mathbf{a}=\begin{pmatrix}1\\1\\1\end{pmatrix}\ \overrightarrow{AC}=\mathbf{c}-\mathbf{a}=\begin{pmatrix}-1\\-2\\1\end{pmatrix}\ and\ \overrightarrow{AD}=\mathbf{d}-\mathbf{a}=\begin{pmatrix}3\\4\\7\end{pmatrix}$$

$$\overrightarrow{AB}.\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)=\begin{vmatrix}1&1&1\\-1&-2&1\\3&4&7\end{vmatrix}=(-14-4)-(-7-3)+(-4+6)=-18+10+2=-6$$

$V=\left|\overrightarrow{AB}.\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)\right|=|-6|$ $\qquad V=6$

b) *i*) A direction vector of the line BD is the vector $\overrightarrow{BD}=\mathbf{d}-\mathbf{b}=\begin{pmatrix}2\\3\\6\end{pmatrix}$

The direction ratios are 2:3:6

ii) $\left|\overrightarrow{BD}\right|=\sqrt{2^2+3^2+6^2}=\sqrt{49}=7$

The direction cosines are $\frac{2}{7},\frac{3}{7},\frac{6}{7}$

Question 6:	Exam report
$a)\ \det(\mathbf{M}) = \begin{vmatrix} -1 & 4 \\ -1 & 3 \end{vmatrix} = -3+4=1$ The area scale factor is 1. The area is invariant under T $b)\ \det(\mathbf{M}-\lambda\mathbf{I}) = \begin{vmatrix} -1-\lambda & 4 \\ -1 & 3-\lambda \end{vmatrix} = (-1-\lambda)(3-\lambda)+4=0$ $\Leftrightarrow \lambda^2-2\lambda+1=0 \Leftrightarrow (\lambda-1)^2=0 \qquad \lambda=1$ The unique eigenvalue is 1 To find the corresponding eigenvector, let's solve $(\mathbf{M}\text{-}\lambda\mathbf{I})\begin{pmatrix} x \\ y \end{pmatrix}=0$ $\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\times\begin{pmatrix} x \\ y \end{pmatrix}=0 \Leftrightarrow \begin{cases} -2x+4y=0 \\ -x+2y=0 \end{cases}$ *a solution is* $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ The line, going through O, with direction vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is invariant by T. c) Consider a point belonging to the line $y=\frac{1}{2}x+k$: $\begin{pmatrix} x \\ \frac{1}{2}x+k \end{pmatrix}$ The image through T is $\mathbf{M}\times\begin{pmatrix} x \\ \frac{1}{2}x+k \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}\times\begin{pmatrix} x \\ \frac{1}{2}x+k \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ $so \begin{cases} x'=-x+2x+4k \\ y'=-x+\frac{3}{2}x+3k \end{cases} \Leftrightarrow \begin{cases} x'=x+4k \\ y'=\frac{1}{2}x+3k \end{cases} \Leftrightarrow \begin{cases} x=x'-4k \\ y'=\frac{1}{2}(x'-4k)+3k \end{cases}$ This gives $y'=\frac{1}{2}x'+k$ $d)$ T is a shear in the direction of the line $y=\frac{1}{2}x$ *mapping* $(1,0)$ *to* $(-1,-1)$ *and* $(0,1)$ *to* $(4,3)$.	This question required several different ideas. In part (a), it was essential to use the word "area" to describe the significance of det **M** in relation to T, although responses were much more frequently appropriate in this respect than had been the case the last time such a question appeared. A similar type of comment was required at the end of part (b). In part (c), those who parametrised the line as (*x*, ½*x* + *k*) generally coped very well, although a significant number lost the final mark by not showing the necessary working to support the given answer carefully, as opposed to merely stating what had been stated in the question. Responses to part (d) proved extremely puzzling: candidates were told that the transformation was a shear, so it was rather strange to find so many including descriptions of (often multiple) stretches, enlargements, rotations and reflections. Candidates are encouraged to look at the mark scheme to find ways in which shears can be described.

Question 7:

$a)\ i)\ \mathbf{D}=\begin{pmatrix}3 & 0\\ 0 & -3\end{pmatrix},\ \mathbf{U}=\begin{pmatrix}1 & 1\\ 2 & 4\end{pmatrix}.\ Det(\mathbf{U})=4-2=2$

$so\ \ \mathbf{U}^{-1}=\frac{1}{2}\begin{pmatrix}4 & -1\\ -2 & 1\end{pmatrix}$

$ii)\ \mathbf{DU}^{-1}=\frac{1}{2}\begin{pmatrix}3 & 0\\ 0 & -3\end{pmatrix}\begin{pmatrix}4 & -1\\ -2 & 1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}12 & -3\\ 6 & -3\end{pmatrix}$

$\mathbf{UDU}^{-1}=\frac{1}{2}\begin{pmatrix}1 & 1\\ 2 & 4\end{pmatrix}\begin{pmatrix}12 & -3\\ 6 & -3\end{pmatrix}=\frac{1}{2}\begin{pmatrix}18 & -6\\ 48 & -18\end{pmatrix}$

$\mathbf{M}=\begin{pmatrix}9 & -3\\ 24 & -9\end{pmatrix}$

$b)\ \mathbf{D}^n=\begin{pmatrix}3^n & 0\\ 0 & (-3)^n\end{pmatrix}\ so\ when\ n\ is\ even\ \ \mathbf{D}^n=\begin{pmatrix}3^n & 0\\ 0 & 3^n\end{pmatrix}=3^n\mathbf{I}$

$Hence\ \mathbf{M}^n=\mathbf{UD}^n\mathbf{U}^{-1}=\mathbf{U}\times 3^n\mathbf{I}\times\mathbf{U}^{-1}=3^n\mathbf{UIU}^{-1}=3^n\mathbf{UU}^{-1}=3^n\mathbf{I}$

- If n is odd, Let's write $\mathbf{M}^n=\mathbf{M}^{n-1}\times\mathbf{M}$

$then\ n-1\ is\ even\ so\ \mathbf{M}^{n-1}=3^{n-1}\mathbf{I}\quad and\ \ \mathbf{M}^n=3^{n-1}\mathbf{I}\times\mathbf{M}=3^{n-1}\mathbf{M}$

Exam report

The more confident candidates found this question to be a source of easy marks. The real difficulties arose in part (b) due to a widespread inability to use brackets and deal with minus signs. When candidates wrote -3^n when they actually meant $(-3)^n$, and it was necessary to read on to see what they eventually did with the terms involving powers of (– 3). The standard approach appears in the mark scheme, though the alternative algebraic ones also appeared quite often, and these were much more pleasing to the eye. For n even, noting that $\mathbf{D}^n = 3^n\,\mathbf{I}$, we have $\mathbf{M}^n = 3^n\,\mathbf{U}\,\mathbf{I}\,\mathbf{U}^{-1}$, which gives the required result both obviously and quickly. However, a small number attempted this approach, but wrote something along the lines of $\mathbf{M}^n = \mathbf{U}\,\mathbf{D}^n\,\mathbf{U}^{-1} = \mathbf{D}^n\,\mathbf{U}\,\mathbf{U}^{-1} = \mathbf{D}^n$ first. This is wrong and scored 0/3. For n odd, a similar 'shortcut' arises via $\mathbf{M}^n=3^n\mathbf{U}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathbf{U}^{-1}$. However, a very small number of candidates are to be highly commended for the following insightful, approach: if n is odd, then $n - 1$ is even, so that $\mathbf{M}^n = \mathbf{M}^{n-1}\,\mathbf{M} = (3^{n-1}\,\mathbf{I})\,(\mathbf{M})$ using the result of part (b)(i) $= 3^{n-1}\,\mathbf{M}$, as required.

Question 8:

$a)\ \det(\mathbf{M})=\begin{vmatrix}a & b & c\\ c & a & b\\ b & c & a\end{vmatrix}=a\left(a^2-bc\right)-b(ca-b^2)+c(c^2-ab)$

$=a^3-abc-abc+b^3+c^3-abc$

$=a^3+b^3+c^3-3abc$

$b)\ \mathbf{MN}=\begin{pmatrix}a & b & c\\ c & a & b\\ b & c & a\end{pmatrix}\times\begin{pmatrix}d & e & f\\ f & d & e\\ e & f & d\end{pmatrix}=\begin{pmatrix}ad+bf+ce & ae+bd+cf & af+be+dc\\ cd+af+be & ce+ad+bf & cf+ae+db\\ db+cf+ae & be+cd+af & bf+ce+da\end{pmatrix}$

$c)\ \det(\mathbf{MN})=\det(\mathbf{M})\times\det(\mathbf{N})$

$=(a^3+b^3+c^3-3abc)(d^3+e^3+f^3-3def)$

$Let\ ad+bf+ce=x\ ,\ ae+bd+cf=y\ and\ af+be+dc=z$

$then\ \mathbf{MN}=\begin{pmatrix}x & y & z\\ z & x & y\\ y & z & x\end{pmatrix}\ \ and\ \ \det(\mathbf{MN})=x^3+y^3+z^3-3xyz$

Exam report

Most candidates made a good attempt at the first two parts and only a few got really 'bogged down' in part (c). It was necessary to spotting that det (**MN**) = det (**M**) det (**N**), and these final two marks of the paper were nicely discriminating of the more able candidates' flexibility. Quite a few other candidates managed to see what was going on but didn't quite realise how to explain why the result arose or to identify correctly the x, y and z referred to.

Grade boundaries							
Grade			A	B	C	D	E
Mark	Max 75		60	52	44	36	28