General Certificate of Education
June 2008
Advanced Level Examination

MATHEMATICS **MFP4**
Unit Further Pure 4

Wednesday 21 May 2008 1.30 pm to 3.00 pm

For this paper you must have:
- a 12-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP4.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{bmatrix} 7 & 12 \\ 12 & 0 \end{bmatrix}$. *(6 marks)*

2 The vectors **a**, **b** and **c** are given by

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{c} = -2\mathbf{i} + t\mathbf{j} + 6\mathbf{k}$$

where t is a scalar constant.

(a) Determine, in terms of t where appropriate:

(i) $\mathbf{a} \times \mathbf{b}$; *(2 marks)*

(ii) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$; *(2 marks)*

(iii) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. *(2 marks)*

(b) Find the value of t for which **a**, **b** and **c** are linearly dependent. *(2 marks)*

(c) Find the value of t for which **c** is parallel to $\mathbf{a} \times \mathbf{b}$. *(2 marks)*

3 The matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 4 & 3 & k \end{bmatrix}$, where k is a constant.

Determine, in terms of k where appropriate:

(a) $\det \mathbf{A}$; *(2 marks)*

(b) $\mathbf{A}^{-1}$. *(5 marks)*

4 Two planes have equations

$$\mathbf{r} \cdot \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix} = 12 \quad \text{and} \quad \mathbf{r} \cdot \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = 7$$

(a) Find, to the nearest 0.1°, the acute angle between the two planes. *(4 marks)*

(b) (i) The point $P(0, a, b)$ lies in both planes. Find the value of a and the value of b. *(3 marks)*

(ii) By using a vector product, or otherwise, find a vector which is parallel to both planes. *(2 marks)*

(iii) Find a vector equation for the line of intersection of the two planes. *(2 marks)*

5 A plane transformation is represented by the 2×2 matrix $\mathbf{M}$. The eigenvalues of $\mathbf{M}$ are 1 and 2, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

(a) State the equations of the invariant lines of the transformation and explain which of these is also a line of invariant points. *(3 marks)*

(b) The diagonalised form of $\mathbf{M}$ is $\mathbf{M} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

(i) Write down a suitable matrix $\mathbf{D}$ and the corresponding matrix $\mathbf{U}$. *(2 marks)*

(ii) Hence determine $\mathbf{M}$. *(4 marks)*

(iii) Show that $\mathbf{M}^n = \begin{bmatrix} 1 & \mathrm{f}(n) - 1 \\ 0 & \mathrm{f}(n) \end{bmatrix}$ for all positive integers n, where $\mathrm{f}(n)$ is a function of n to be determined. *(3 marks)*

Turn over for the next question

6 Three planes have equations

$$\begin{aligned} x + y - 3z &= b \\ 2x + y + 4z &= 3 \\ 5x + 2y + az &= 4 \end{aligned}$$

where a and b are constants.

(a) Find the coordinates of the single point of intersection of these three planes in the case when $a = 16$ and $b = 6$. *(5 marks)*

(b) (i) Find the value of a for which the three planes do not meet at a single point. *(3 marks)*

(ii) For this value of a, determine the value of b for which the three planes share a common line of intersection. *(5 marks)*

7 A transformation T of three-dimensional space is given by the matrix $\mathbf{W} = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$.

(a) (i) Evaluate $\det \mathbf{W}$, and describe the geometrical significance of the answer in relation to T. *(2 marks)*

(ii) Determine the eigenvalues of $\mathbf{W}$. *(6 marks)*

(b) The plane H has equation $\mathbf{r} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 0$.

(i) Write down a cartesian equation for H. *(1 mark)*

(ii) The point P has coordinates (a, b, c). Show that, whatever the values of a, b and c, the image of P under T lies in H. *(4 marks)*

8 By considering the determinant

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}$$

show that $(x + y + z)$ is a factor of $x^3 + y^3 + z^3 - kxyz$ for some value of the constant k to be determined. *(3 marks)*

END OF QUESTIONS

Key to mark scheme and abbreviations used in marking

M	mark is for method
m or dM	mark is dependent on one or more M marks and is for method
A	mark is dependent on M or m marks and is for accuracy
B	mark is independent of M or m marks and is for method and accuracy
E	mark is for explanation

√ or ft or F	follow through from previous incorrect result	MC	mis-copy
CAO	correct answer only	MR	mis-read
CSO	correct solution only	RA	required accuracy
AWFW	anything which falls within	FW	further work
AWRT	anything which rounds to	ISW	ignore subsequent work
ACF	any correct form	FIW	from incorrect work
AG	answer given	BOD	given benefit of doubt
SC	special case	WR	work replaced by candidate
OE	or equivalent	FB	formulae book
A2,1	2 or 1 (or 0) accuracy marks	NOS	not on scheme
$-x$ EE	deduct x marks for each error	G	graph
NMS	no method shown	c	candidate
PI	possibly implied	sf	significant figure(s)
SCA	substantially correct approach	dp	decimal place(s)

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded. However, there are situations in some units where part marks would be appropriate, particularly when similar techniques are involved. Your Principal Examiner will alert you to these and details will be provided on the mark scheme.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

MFP4

Q	Solution	Marks	Total	Comments
1	Attempt at char eqn $\lambda^2 - 7\lambda - 144 = 0$	M1		Any suitable method Ignore missing "= 0"
	Solving quadratic to find evals	M1		Any method
	$\lambda = 16$ or -9	A1		CAO
	$\lambda = 16 \Rightarrow -9x + 12y = 0 \Rightarrow y = \frac{3}{4}x$	M1		Either λ substituted back
	$\Rightarrow$ evecs $\alpha\begin{bmatrix}4\\3\end{bmatrix}$	A1		CAO (for any non-zero α)
	$\lambda = -9 \Rightarrow 16x + 12y = 0 \Rightarrow y = -\frac{4}{3}x$			
	$\Rightarrow$ evecs $\beta\begin{bmatrix}3\\-4\end{bmatrix}$	A1	6	CAO (for any non-zero β)
	Total		**6**	
2(a)(i)	$\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 3\\ 2 & 1 & 2\end{vmatrix} = \begin{bmatrix}1\\4\\-3\end{bmatrix}$	M1 A1	 2	Genuine vector product attempt CAO
(ii)	$(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c} = \begin{bmatrix}1\\4\\-3\end{bmatrix}\bullet\begin{bmatrix}-2\\t\\6\end{bmatrix} = 4t - 20$	M1 A1	 2	Must get a scalar answer ft
(iii)	$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 4 & -3\\ -2 & t & 6\end{vmatrix} = \begin{bmatrix}3t+24\\0\\t+8\end{bmatrix}$	M1 A1	 2	Either using (a)(i) or starting again CAO
(b)	$(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c} = 0 \Rightarrow t = 5$	M1A1	2	ft from (a)(ii)
(c)	$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{0}$ or $\mathbf{c}$ = mult. of $(\mathbf{a}\times\mathbf{b})$	M1		Use of any non-zero row to find some value of t
	$\Rightarrow t = -8$	A1	2	CAO – allow unseen check
	Total		**10**	
3(a)	Det $\mathbf{A} = k + 3 + 12 - 4 - 9 - k = 2$	M1 A1	 2	 CAO
(b)	$\mathbf{A}^{-1} = \frac{1}{\text{Det A}}(\text{adj }\mathbf{A})$	B1 M1		Correct use of the determinant (any value) Attempt at matrix of cofactors
	$= \frac{1}{2}\begin{bmatrix}k-9 & 3-k & 2\\ 12-k & k-4 & -2\\ -1 & 1 & 0\end{bmatrix}$	M1 A1 A1	 5	Use of transposition **and** signs At least 5 entries correct (even if 2nd M1 not earned) CAO – ft det only
	Total		**7**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
4(a)	Use of $\cos\theta = \dfrac{\text{scalar product}}{\text{product of moduli}}$	M1		**Must** be $\begin{bmatrix}2\\1\\4\end{bmatrix}$ and $\begin{bmatrix}5\\1\\-1\end{bmatrix}$ "$\sin\theta =$" scores M0 at this stage
	Numerator = 7 Denominator = $\sqrt{21}\sqrt{27}$	B1,B1		Allow denominator unsimplified
	$\theta = 72.9^{\circ}$	A1	4	CAO (but A0 if candidate proceeds to find its complement)
(b)(i)	$a + 4b = 7$ and $a - b = 12$	B1		At least one correctly stated
		M1		Solving simultaneously
	$a = 11$ and $b = -1$	A1	3	CAO
(ii)	$\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\5 & 1 & -1\\2 & 1 & 4\end{bmatrix} = \begin{bmatrix}5\\-22\\3\end{bmatrix}$	M1		For any valid, complete method for finding a suitable direction **vector**, eg finding a 2nd common point, eg (2½ , 0, ½) or (1⅔, 3⅔, 0), and then **dv** = difference
		A1	2	CAO
(iii)	$\mathbf{r} = \begin{bmatrix}0\\11\\-1\end{bmatrix} + \lambda\begin{bmatrix}5\\-22\\3\end{bmatrix}$	M1		**Must** be a line equation and use their (b)(ii)
	or other equivalent line form eg $(\mathbf{r} - \mathbf{a}) \times \mathbf{d} = \mathbf{0}$	A1	2	ft their (b)(i) point, or any other correct point on the line A0 if no **r** = or l = etc
	Total		**11**	
5(a)	$y = 0$ (or "x-axis") and $y = x$	B1,B1		or $\mathbf{r} = a\begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{r} = b\begin{bmatrix}1\\1\end{bmatrix}$
	$y = 0$ is a line of invariant points since $\lambda = 1$	B1	3	Allow if proven from $(x', y') = (x, y)$ or ft from their line corresponding to $\lambda = 1$
(b)(i)	$\mathbf{D} = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$, $\mathbf{U} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$,	B1,B1	2	ft **U** from **D**
(ii)	$\mathbf{U}^{-1} = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$	B1		ft from **U** (provided non-singular)
	$\mathbf{M} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$	M1		Attempt
	$= \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}$ or $\begin{bmatrix}1 & 2\\0 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$	A1		ft first multiplication
	$= \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}$	A1	4	CAO NMS $\Rightarrow$ 0

MFP4 (cont)

Q	Solution	Marks	Total	Comments
(iii)	$\mathbf{D}^n = \begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix}$	B1		Noted or used
	$\mathbf{M}^n = \mathbf{U}\,\mathbf{D}^n\,\mathbf{U}^{-1}$	M1		Used; **must** actually do some multiplying
	$= \begin{bmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{bmatrix}$	A1	3	
	Total		**12**	
6(a)	eg **(2) – (1)** $\Rightarrow x + 7z = -3$	M1A1		Eliminating first variable
	(3) $- 2 \times$ **(2)** $\Rightarrow x + 8z = -2$	A1		
	Solving 2×2 system	M1		
	$x = -10$, $y = 19$, $z = 1$	A1	5	
(b)(i)	$\begin{vmatrix} 1 & 1 & -3 \\ 2 & 1 & 4 \\ 5 & 2 & a \end{vmatrix} = 15 - a$	B1		Determinant
	Setting = to zero and solving for a	M1		**Must** get a numerical answer
	$a = 15$	A1	3	ft
(ii)	$x + y - 3z = b$ $2x + y + 4z = 3$ $5x + 2y + 15z = 4$			
	eg **(2) – (1)** $\Rightarrow x + 7z = 3 - b$	M1A1		Eliminating first variable
	(3) $- 2 \times$ **(2)** $\Rightarrow x + 7z = -2$	A1		
	Equating the two RHSs	M1		
	$b = 5$	A1	5	CAO
				NB Eliminating x : $-y + 10z = 3 - 2b$ $-3y + 30z = 4 - 5b$ $-y + 10z = -7$
				NB Eliminating z : $10x + 7y = 4b + 9$ $10x + 7y = 5b + 4$ $10x + 7y = 29$
	Total		**13**	
6(a)	**Alternate Schemes** **Cramer's Rule**			
	$\Delta = \begin{vmatrix} 1 & 1 & -3 \\ 2 & 1 & 4 \\ 5 & 2 & 16 \end{vmatrix}$, $\Delta_x = \begin{vmatrix} 6 & 1 & -3 \\ 3 & 1 & 4 \\ 4 & 2 & 16 \end{vmatrix}$, $\Delta_y = \begin{vmatrix} 1 & 6 & -3 \\ 2 & 3 & 4 \\ 5 & 4 & 16 \end{vmatrix}$, $\Delta_z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 1 & 3 \\ 5 & 2 & 4 \end{vmatrix}$	M1		Attempt at any two
	$= -1, 10, -19$ and -1 respectively	A1		Any one correct
	$x = \dfrac{\Delta_x}{\Delta}$, $y = \dfrac{\Delta_y}{\Delta}$, $z = \dfrac{\Delta_z}{\Delta}$	M1		At least one attempted numerically
	$x = -10$, $y = 19$, $z = 1$	A1		Any 2 correct ft
		A1	(5)	All 3 correct CAO

MFP4 (cont)

Q	Solution	Marks	Total	Comments
6(a)	**Inverse matrix method**			
	$C^{-1} = \frac{1}{-1}\begin{bmatrix} 8 & -22 & 7 \\ -12 & 31 & -10 \\ -1 & 3 & -1 \end{bmatrix}$	M1 A1		M0 if no inverse matrix is given
	$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = C^{-1}\begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \\ 1 \end{bmatrix}$	M1 A1 A1	(5)	Any 2 correct ft All 3 correct CAO
6(all)	$\left[\begin{array}{ccc\|c} 1 & 1 & -3 & b \\ 2 & 1 & 4 & 3 \\ 5 & 2 & a & 4 \end{array}\right]$			
	$\rightarrow \left[\begin{array}{ccc\|c} 1 & 1 & -3 & b \\ 0 & -1 & 10 & 3-2b \\ 0 & -3 & a+15 & 4-5b \end{array}\right]$			$R_2' = R_2 - 2R_1$ $R_3' = R_3 - 5R_1$
	$\rightarrow \left[\begin{array}{ccc\|c} 1 & 1 & -3 & b \\ 0 & 1 & -10 & 2b-3 \\ 0 & 0 & a-15 & b-5 \end{array}\right]$		(4)	$R_3' = R_3 + 3R_2$
	(b)(i) For non-unique solutions, $a = 15$		(2)	
	(ii) For consistency, $4 - 5b = 3(3 - 2b) \Rightarrow b = 5$		(2)	
	(a) When $a = 16, b = 6$ $\left[\begin{array}{ccc\|c} 1 & 1 & -3 & 6 \\ 0 & 1 & -10 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right]$ $\Rightarrow z = 1,\ y = 19,\ x = -10$		(5)	
7(a)(i)	det **W** = 0 Transformed shapes have zero volume	B1 B1	2	Or equivalent statement ft volume statements
(ii)	Char eqn is $\lambda^3 - 4\lambda^2 + 4\lambda = 0$	M1A3		One A mark for each coefficient (not the λ^3)
	Solving the cubic eqn $\lambda = 0, 2, (2)$	M1 A1	6	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
7(a)(ii)	**Alternative:**			
	Det ($\mathbf{W} - \lambda\mathbf{I}$) = $\begin{vmatrix} 3-\lambda & -1 & 1 \\ 2 & -\lambda & 2 \\ -1 & 1 & 1-\lambda \end{vmatrix}$			
	$= \begin{vmatrix} 2-\lambda & 0 & 2-\lambda \\ 2 & -\lambda & 2 \\ -1 & 1 & 1-\lambda \end{vmatrix}$	M1		Use of R/C ops. $R_1' \to R_1 + R_3$
	$= (2-\lambda)\begin{vmatrix} 1 & 0 & 1 \\ 2 & -\lambda & 2 \\ -1 & 1 & 1-\lambda \end{vmatrix}$	A1		Factor of $(2 - \lambda)$ correctly extracted
	$= (2-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 2 & -\lambda & 0 \\ -1 & 1 & 2-\lambda \end{vmatrix}$			$C_1' \to C_3 - C_1$
	$= (2-\lambda)^2\begin{vmatrix} 1 & 0 & 0 \\ 2 & -\lambda & 0 \\ -1 & 1 & 1 \end{vmatrix}$	A1A1		
	$= (2-\lambda)^2(-\lambda)$	M1		Complete factorisation attempt
	giving eigenvalues 0 and 2 (twice)	A1	(6)	
(b)(i)	$x - y + z = 0$	B1	1	
(ii)	$\begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3a-b+c \\ 2a+2c \\ -a+b+c \end{bmatrix}$	M1A1		
	$x' - y' + z' = 3a - b + c - 2a - 2c - a + b + c$	M1		
	$= 0 \Rightarrow P'$ in H also	A1	4	Shown carefully
	Total		**13**	
8	Expanding fully: $\Delta = x^3 + y^3 + z^3 - 3xyz$	B1		
	Using row/column operations: eg $R_1' = R_1 + (R_2 + R_3)$	M1		
	$\Rightarrow \Delta = (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ y & z & x \end{vmatrix}$	A1	3	With conclusion that $(x + y + z)$ is a factor of the required expression when $k = 3$
	NB Any line of argument that leads correctly from $(x + y + z)\,f(x, y, z)$ to $x^3 + y^3 + z^3 - 3xyz$ scores full marks			
	Total		**3**	
	TOTAL		**75**	

AQA – Further pure 4 – Jun 2008 – Answers

Question 1:	Exam report
Let's call $\mathbf{M}$ the matrix $\begin{pmatrix} 7 & 12 \\ 12 & 0 \end{pmatrix}$ To find the eigen values, let's solve $\det(\mathbf{M}-\lambda\mathbf{I})=0$ $\det(\mathbf{M}-\lambda\mathbf{I})=\begin{vmatrix} 7-\lambda & 12 \\ 12 & -\lambda \end{vmatrix}=-\lambda(7-\lambda)-144$ $=\lambda^2-7\lambda-144$ $\det(\mathbf{M}-\lambda\mathbf{I})=0 \Leftrightarrow \lambda^2-7\lambda-144=0$ $(\lambda-16)(\lambda+9)=0$ The eigenvalues are $\lambda=16$ and $\lambda=-9$ • The eigenvector for $\lambda=16$ is obtained by solving $(\mathbf{M}-16\mathbf{I})\begin{pmatrix} x \\ y \end{pmatrix}=0$ i.e $\begin{cases} -9x+12y=0 \\ 12x-16y=0 \end{cases} \Leftrightarrow \begin{cases} 3x-4y=0 \\ 3x-4y=0 \end{cases}$ An eigenvector is $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ • The eigenvector for $\lambda=-9$ is obtained by solving $(\mathbf{M}+9\mathbf{I})\begin{pmatrix} x \\ y \end{pmatrix}=0$ i.e $\begin{cases} 16x+12y=0 \\ 12x+9y=0 \end{cases} \Leftrightarrow \begin{cases} 4x+3y=0 \\ 4x+3y=0 \end{cases}$ An eigenvector is $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$	This was a routine, straightforward question. A few candidates had difficulty turning a Cartesian line equation into an eigenvector, eg 3x = 4y became the vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ rather than $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

Question 2:	Exam report
$a)\, i)\, \mathbf{a}\times\mathbf{b}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 1 & 2 \end{vmatrix}=\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}\mathbf{i}-\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix}\mathbf{j}+\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}\mathbf{k}$ $\mathbf{a}\times\mathbf{b}=\mathbf{i}+4\mathbf{j}-3\mathbf{k}$ $ii)\, (\mathbf{a}\times\mathbf{b}).\mathbf{c}=\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}.\begin{pmatrix} -2 \\ t \\ 6 \end{pmatrix}=-2+4t-18=4t-20$ $iii)\, (\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -3 \\ -2 & t & 6 \end{vmatrix}=\begin{vmatrix} 4 & -3 \\ t & 6 \end{vmatrix}\mathbf{i}-\begin{vmatrix} 1 & -3 \\ -2 & 6 \end{vmatrix}\mathbf{j}+\begin{vmatrix} 1 & 4 \\ -2 & t \end{vmatrix}\mathbf{k}$ $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(24+3t)\mathbf{i}+(t+8)\mathbf{k}$ $b)$ $\mathbf{a},\mathbf{b}$ *and* $\mathbf{c}$ are linearly dependent when $(\mathbf{a}\times\mathbf{b}).\mathbf{c}=0$ this occurs when $4t-20=0$ $t=5$ $c)$ $\mathbf{c}$ is parallel to $\mathbf{a}\times\mathbf{b}$ when $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=0$ this occurs when $24+3t=0$ and $t+8=0$ $t=-8$	This was another routine, straightforward question. Most candidates realised that the answers required in parts (b) and (c) came from those found in part (a). Others started again, working out, for instance, that **c** = – 2(**a** × **b**) in order to find the value of t required in part (c).

Question 3:

$$a)\ \det(\mathbf{A}) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 4 & 3 & k \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & k \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 4 & k \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} \quad \det(\mathbf{A}) = k - 9 - k + 12 + 3 - 4 = 2$$

$$b)\ cofactor(\mathbf{A}) = \begin{pmatrix} k-9 & 12-k & -1 \\ 3-k & k-4 & 1 \\ 2 & -2 & 0 \end{pmatrix};\ Adj(\mathbf{A}) = cofactor(\mathbf{A})^T = \begin{pmatrix} k-9 & 3-k & 2 \\ 12-k & k-4 & -2 \\ -1 & 1 & 0 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{2} Adj(\mathbf{A}) = \frac{1}{2}\begin{pmatrix} k-9 & 3-k & 2 \\ 12-k & k-4 & -2 \\ -1 & 1 & 0 \end{pmatrix}$$

Exam report

Almost all candidates did very well at finding the inverse matrix using the ‘transposed matrix of co-factors’ approach, and only a small proportion forgot the alternating signs and/or the transposition.

Question 4:

$a)$ A normal vector to the plane P_1 *is* $\mathbf{n}_1 = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$

A normal vector to the plane P_2 *is* $\mathbf{n}_2 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$

Let's work out the angle θ between these two vectors:

$$\mathbf{n}_1.\mathbf{n}_2 = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}.\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = 10 + 1 - 4 = 7 \qquad |\mathbf{n}_1| = \sqrt{25+1+1} = \sqrt{27}$$

$$|\mathbf{n}_2| = \sqrt{4+1+16} = \sqrt{21} \qquad so \quad Cos\theta = \frac{\mathbf{n}_1.\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{7}{\sqrt{27}\sqrt{21}} = \frac{\sqrt{7}}{9}$$

$$\theta = Cos^{-1}\left(\frac{\sqrt{7}}{9}\right) = 72.9^o \quad (acute)$$

$b)\ i)\ P(0,a,b)$ lies on both plane so $\begin{pmatrix} 0 \\ a \\ b \end{pmatrix}.\begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} = 12$ *and* $\begin{pmatrix} 0 \\ a \\ b \end{pmatrix}.\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = 7$

This gives the following equations: $\begin{cases} a - b = 12 \\ a + 4b = 7 \end{cases} \Leftrightarrow \begin{cases} 5b = -5 \\ a = 12 + b \end{cases} \Leftrightarrow \begin{cases} b = -1 \\ a = 11 \end{cases}$

The point P has coordinates $(0,11,-1)$

$ii)$ A line parallel to both planes has the direction vector perpendicular to both $\mathbf{n}_1$ *and* $\mathbf{n}_2$.

A vector satisfying this condition is $\mathbf{n}_1 \times \mathbf{n}_2$:

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 1 & -1 \\ 2 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 5 & -1 \\ 2 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 5 & 1 \\ 2 & 1 \end{vmatrix}\mathbf{k} = 5\mathbf{i} - 22\mathbf{j} + 3\mathbf{k}$$

$iii)$ P(0,11,-1) belongs to both plane so belongs the the line of intersection and a direction vector of this line is $5\mathbf{i} - 22\mathbf{j} + 3\mathbf{k}$

An equation vector of the line of intersection is $\mathbf{r} = \begin{pmatrix} 0 \\ 11 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -22 \\ 3 \end{pmatrix}$

$$or\ if\ you\ prefer: \left(\mathbf{r} - \begin{pmatrix} 0 \\ 11 \\ -1 \end{pmatrix}\right) \times \begin{pmatrix} 5 \\ -22 \\ 3 \end{pmatrix} = 0$$

Exam report

This proved to be another accessible question, with the question’s structuring helping candidates. Only a few opted to start again for part (b)(iii), for instance.

Question 5:

$a)$ The invariant lines go through the origin

with direction vector $\begin{pmatrix}1\\0\end{pmatrix}$ *and* $\begin{pmatrix}1\\1\end{pmatrix}$ *respectively*

An equation of these lines are $y = x$ *and* $y = 0$

The line of invariant points is $y = 0$ corresponding the the eigenvalue $\lambda = 1$.

$b)\ \mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$

$i)\ \mathbf{D} = \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}$ *and* $\mathbf{U} = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$

$ii)\ \det(\mathbf{U}) = \begin{vmatrix}1 & 1\\0 & 1\end{vmatrix} = 1$ *and* $\mathbf{U}^{-1} = \begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}$

$$\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1} = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\0 & 2\end{pmatrix} = \begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}$$

$$iii)\ \mathbf{M}^n = \mathbf{U}\mathbf{D}^n\mathbf{U}^{-1} = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1^n & 0\\0 & 2^n\end{pmatrix}\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\0 & 2^n\end{pmatrix} = \begin{pmatrix}1 & 2^n - 1\\0 & 2^n\end{pmatrix}$$

$f(n) = 2^n$

Exam report

Most difficulties with this question arose in part (a), where candidates had to consider the meaning and relevance of the information given to them in the question. The most commonly lost mark was in the widespread failure to point out that the line of invariant points was flagged up by the fact that the eigenvalue corresponding to the *x*-axis was 1. In general, most other marks lost on this question arose as a result of minor slips and oversights in the working. A few candidates didn't seem to know how to find $\mathbf{M}^n$.

Question 6:

$a)$ *for* $a = 16$ *and* $b = 6$, the system of equation becomes:

$$\begin{cases} x + y - 3z = 6 & l_1 \\ 2x + y + 4z = 3 & l_2 \\ 5x + 2y + 16z = 4 & l_3 \end{cases} \Leftrightarrow \begin{cases} x + y - 3z = 6 & \\ x \quad + 7z = -3 & l_2 - l_1 = l_2' \\ 3x \quad + 22z = -8 & l_3 - 2l_1 = l_3' \end{cases} \Leftrightarrow \begin{cases} x + y - 3z = 6 & \\ x + 7z = -3 & \\ z = 1 & l_3' - 3l_2' \end{cases}$$

$$\Leftrightarrow \begin{cases} y = -x + 3z + 6 = 19 \\ x = -7z - 3 = -10 \\ z = 1 \end{cases}$$

The three planes intersect at $P(-10, 19, 1)$

$b)\ i)$ The system of equations does not have a unique solution when the determinant of the associated matrix is equal to 0:

$$\begin{vmatrix}1 & 1 & -3\\2 & 1 & 4\\5 & 2 & a\end{vmatrix} = \begin{vmatrix}1 & 4\\2 & a\end{vmatrix} - \begin{vmatrix}2 & 4\\5 & a\end{vmatrix} - 3\begin{vmatrix}2 & 1\\5 & 2\end{vmatrix} = a - 8 - 2a + 20 - 12 + 15 = 15 - a$$

$\det = 0$ *for* $a = 15$

$ii)$ For $a = 15$, we want the system to be consistent.

$$\begin{cases} x + y - 3z = b & l_1 \\ 2x + y + 4z = 3 & l_2 \\ 5x + 2y + 15z = 4 & l_3 \end{cases} \Leftrightarrow \begin{cases} x + y - 3z = 6 & \\ x \quad + 7z = 3 - b & l_2 - l_1 = l_2' \\ 3x \quad + 21z = 4 - 2b & l_3 - 2l_1 = l_3' \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y - 3z = 6 & \\ 3x + 21z = 3(3 - b) & 3l_2' \\ 3x + 21z = 4 - 2b & \end{cases}$$

The system is consistent if and only if

$3(3 - b) = 4 - 2b$

$9 - 3b = 4 - 2b$ $\qquad b = 5$

Exam report

Despite a pleasing array of higher-powered techniques on display in solutions here, the lowtech method of first reducing the 3 × 3 system to a 2 × 2 system is by far the easiest approach, and the one most commonly used by candidates. However, although eliminating the *y*-variable first makes for the least demanding working, candidates' efforts were fairly evenly distributed amongst the three variables. Though this makes no difference in principle, the tendency for weaker candidates not to stop and think about their approach often led to unnecessarily tricky algebra later on in part (b), which then provided more opportunities for errors to occur in the solutions presented.

Question 7:

a) i) $\det(\mathbf{W}) = \begin{vmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{vmatrix} = 3(0-2)+(2+2)+(2-0) = -6+4+2 = 0$

The volume of the transformed shape is 0.

ii) To determine the eigenvalues, we solve $\det(\mathbf{W}-\lambda\mathbf{I})=0$.

$$\det(\mathbf{W}-\lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & -1 & 1 \\ 2 & -\lambda & 2 \\ -1 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 2 & 2-\lambda & 2 \\ -1 & 2-\lambda & 1-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 2 & 2-\lambda & 2 \\ 3 & 0 & 1+\lambda \end{vmatrix}$$

$$\det(\mathbf{W}-\lambda\mathbf{I}) = (2-\lambda)((3-\lambda)(1+\lambda)-3) = (2-\lambda)(-\lambda^2+2\lambda) = -\lambda(2-\lambda)(\lambda-2) = 0$$

The eigenvalues are $\lambda=0$ and $\lambda=2$

b) i) $\mathbf{r}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0 \Leftrightarrow x-y+z=0$

ii) $P(a,b,c)$ *has* position vector $\mathbf{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ *and* its image through T is

$$\mathbf{W}\times\mathbf{p} = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a-b+c \\ 2a+2c \\ -a+b+c \end{pmatrix}$$

The point $P'(3a-b+c\ ,\ 2a+2c\ ,\ -a+b+c)$

let's work out $x_{P'} - y_{P'} + z_{P'} = 3a-b+c-(\ 2a+2c)-a+b+c$

$= 3a-3a-b+b-2c+2c = 0$

The point P' belongs to the plane H for any values of a, b *or* c.

Exam report

Incorporating a 3 × 3 matrix inevitably leads to a higher degree of difficulty in finding the characteristic equation, and this frequently proved to be so for many candidates. Once again, however, it was the introductory remark required to explain the significance of a zero determinant that caused most difficulty. Surprisingly few candidates seemed entirely sure about it, and the most popular suggestion was that det**W** = 0 meant that volumes were unchanged.
However, the majority of responses dwelt exclusively on the theme of areas, which would have been suitable only had **W** been a 2 × 2 matrix. A very few candidates produced a surprising answer, explaining that **W** was a "dimension-destroying (or crushing)" matrix. This gains no credit, unless candidates then go on to explain that 3-d shapes become 2-d ones, or possibly even lines or points, at this stage. A lot of candidates wrote incoherent and meaningless statements: vague statements such as "it is co-planar" don't carry much weight and certainly don't get the marks.

Question 8:

Let's work out the determinant using two methods

Method 1: expanding (developing)

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} = x\begin{vmatrix} x & y \\ z & x \end{vmatrix} - y\begin{vmatrix} z & y \\ y & x \end{vmatrix} + z\begin{vmatrix} z & x \\ y & z \end{vmatrix}$$

$$= x(x^2-zy)-y(zx-y^2)+z(z^2-xy)$$

$$= x^3+y^3+z^3-3xyz$$

Method 2: by combining the lines and columns

Adding the three columns ($C_1' = C_1 + C_2 + C_3$):

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} = \begin{vmatrix} x+y+z & y & z \\ x+y+z & x & y \\ x+y+z & z & x \end{vmatrix} = (x+y+z)\begin{vmatrix} 1 & y & z \\ 1 & x & y \\ 1 & z & x \end{vmatrix}$$

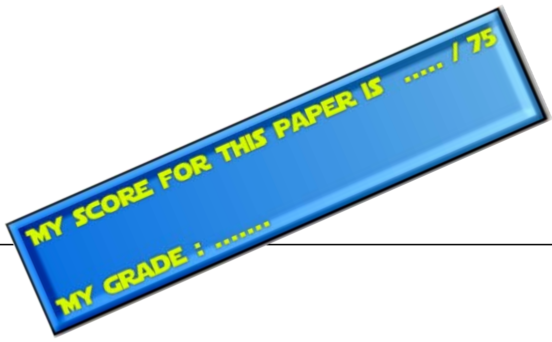
Conclusion : $(x+y+z)$ is a factor of the determinant

$(x+y+z)$ is a factor of $x^3+y^3+z^3-3xyz$

$k=3$

Exam report

Because of its slightly unusual nature, this short question was put last, so that candidates would not spend time they could better employ elsewhere in trying to figure out what to do with it. In the event, it proved far less of a problem than anticipated. The intention was that candidates would expand the determinant fully to gain the expression $x^3 + y^3 + z^3 - 3xyz$ and then use row/column operations to extract the factor of $(x + y + z)$. Most did indeed do so. The mark most commonly lost was the final one of tying the two ends together, which many failed to do, and examiners were quite strict in not giving full marks for solutions which never got round to joining up two otherwise disparate bits of working.



Grade boundaries

Grade			A	B	C	D	E
Mark	Max 75		66	58	51	44	37