General Certificate of Education
June 2007
Advanced Level Examination

MATHEMATICS **MFP4**
Unit Further Pure 4

Friday 22 June 2007 9.00 am to 10.30 am

For this paper you must have:
- a 12-page answer book
- the **blue** AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP4.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 Given that $\mathbf{a} \times \mathbf{b} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ and that $\mathbf{a} \times \mathbf{c} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, determine:

(a) $\mathbf{c} \times \mathbf{a}$; *(1 mark)*

(b) $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$; *(2 marks)*

(c) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c})$; *(2 marks)*

(d) $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{c})$. *(1 mark)*

2 Factorise completely the determinant $\begin{vmatrix} y & x & x+y-1 \\ x & y & 1 \\ y+1 & x+1 & 2 \end{vmatrix}$. *(6 marks)*

3 Three points, A, B and C, have position vectors

$$\mathbf{a} = \begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

respectively.

(a) Using the scalar triple product, or otherwise, show that **a**, **b** and **c** are coplanar. *(2 marks)*

(b) (i) Calculate $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$. *(3 marks)*

(ii) Hence find, to three significant figures, the area of the triangle ABC . *(3 marks)*

4 Consider the following system of equations, where k is a real constant:

$$\begin{aligned} kx &+ 2y &+ z &= 5 \\ x &+ (k+1)y &- 2z &= 3 \\ 2x &- ky &+ 3z &= -11 \end{aligned}$$

(a) Show that the system does not have a unique solution when $k^2 = 16$. *(3 marks)*

(b) In the case when $k = 4$, show that the system is inconsistent. *(4 marks)*

(c) In the case when $k = -4$:

(i) solve the system of equations; *(5 marks)*

(ii) interpret this result geometrically. *(1 mark)*

5 The line l has equation $\mathbf{r} = \begin{bmatrix} 3 \\ 26 \\ -15 \end{bmatrix} + \lambda \begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix}$.

(a) Show that the point $P(-29, 42, -19)$ lies on l. *(1 mark)*

(b) Find:

(i) the direction cosines of l; *(2 marks)*

(ii) the acute angle between l and the z-axis. *(1 mark)*

(c) The plane Π has cartesian equation $3x - 4y + 5z = 100$.

(i) Write down a normal vector to Π. *(1 mark)*

(ii) Find the acute angle between l and this normal vector. *(4 marks)*

(d) Find the position vector of the point Q where l meets Π. *(4 marks)*

(e) Determine the shortest distance from P to Π. *(3 marks)*

Turn over for the next question

6 The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & t \end{bmatrix}$$

(a) Find, in terms of t, the matrices:

(i) **AB**; *(3 marks)*

(ii) **BA**. *(2 marks)*

(b) Explain why **AB** is singular for all values of t. *(1 mark)*

(c) In the case when $t = -2$, show that the transformation with matrix **BA** is the combination of an enlargement, E, and a second transformation, F. Find the scale factor of E and give a full geometrical description of F. *(6 marks)*

7 (a) The matrix $\mathbf{M} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$ represents a shear.

(i) Find $\det \mathbf{M}$ and give a geometrical interpretation of this result. *(2 marks)*

(ii) Show that the characteristic equation of **M** is $\lambda^2 - 2\lambda + 1 = 0$, where λ is an eigenvalue of **M**. *(2 marks)*

(iii) Hence find an eigenvector of **M**. *(3 marks)*

(iv) Write down the equation of the line of invariant points of the shear. *(1 mark)*

(b) The matrix $\mathbf{S} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represents a shear.

(i) Write down the characteristic equation of **S**, giving the coefficients in terms of a, b, c and d. *(2 marks)*

(ii) State the numerical value of $\det \mathbf{S}$ and hence write down an equation relating a, b, c and d. *(2 marks)*

(iii) Given that the only eigenvalue of **S** is 1, find the value of $a + d$. *(2 marks)*

END OF QUESTIONS

Key to mark scheme and abbreviations used in marking

M	mark is for method
m or dM	mark is dependent on one or more M marks and is for method
A	mark is dependent on M or m marks and is for accuracy
B	mark is independent of M or m marks and is for method and accuracy
E	mark is for explanation

√or ft or F	follow through from previous incorrect result	MC	mis-copy
CAO	correct answer only	MR	mis-read
CSO	correct solution only	RA	required accuracy
AWFW	anything which falls within	FW	further work
AWRT	anything which rounds to	ISW	ignore subsequent work
ACF	any correct form	FIW	from incorrect work
AG	answer given	BOD	given benefit of doubt
SC	special case	WR	work replaced by candidate
OE	or equivalent	FB	formulae book
A2,1	2 or 1 (or 0) accuracy marks	NOS	not on scheme
$-x$ EE	deduct x marks for each error	G	graph
NMS	no method shown	c	candidate
PI	possibly implied	sf	significant figure(s)
SCA	substantially correct approach	dp	decimal place(s)

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded. However, there are situations in some units where part marks would be appropriate, particularly when similar techniques are involved. Your Principal Examiner will alert you to these and details will be provided on the mark scheme.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

MFP4

Q	Solution	Marks	Total	Comments
1(a)	$\mathbf{c} \times \mathbf{a} = -\mathbf{a} \times \mathbf{c} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$	B1	1	
(b)	$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$	M1		
	$= 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$	A1	2	
(c)	$(\mathbf{a} \times \mathbf{b}) \,.\, (\mathbf{a} \times \mathbf{c}) = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = -4$	M1		Must attempt to get a scalar
		A1	2	
(d)	$\mathbf{a} \,.\, (\mathbf{a} \times \mathbf{c}) = 0 \quad \left(\text{since } \mathbf{a} \times \mathbf{c} \text{ perp}^{\text{r}} \text{ to } \mathbf{a}\right)$	B1	1	B0 for "0" from invalid working
			6	
2	$\Delta = \begin{vmatrix} y-x & x & x+y-1 \\ x-y & y & 1 \\ y-x & x+1 & 2 \end{vmatrix}$	M1		Attempt at first linear factor, eg $C_1' = C_1 - C_2$
	$= (y-x)\begin{vmatrix} 1 & x & x+y-1 \\ -1 & y & 1 \\ 1 & x+1 & 2 \end{vmatrix}$	A1		For 1st linear factor (Ignore remaining det.)
	$\Delta = (y-x)\begin{vmatrix} 0 & x+y & x+y \\ -1 & y & 1 \\ 1 & x+1 & 2 \end{vmatrix}$	M1		Attempt at second linear factor, eg $R_1' = R_1 + R_2$
	$= (y-x)(y+x)\begin{vmatrix} 0 & 1 & 1 \\ -1 & y & 1 \\ 1 & x+1 & 2 \end{vmatrix}$	A1		For 2nd linear factor (Ignore remaining det.)
	Full expansion	M1		
	$\Delta = (y-x)(y+x)(2-x-y)$	A1	6	Completely correct solution
	Or			
	Setting $y = x \Rightarrow C_1 = C_2 \Rightarrow \Delta = 0$	(M1)		Factor theorem
	$\Rightarrow (y-x)$ a factor of Δ	(A1)		
	Setting $y = -x \Rightarrow R_1 = R_2$	(M1)		
	So that $R_1' = R_1 + R_2 \Rightarrow R_1' = 0$			
	$\Rightarrow \Delta = 0$ and $(y+x)$ a factor of Δ	(A1)		
	Genuine attempt at 3rd factor	(M1)		
	Completely correct solution	(A1)	(6)	
	<u>Additional notes for question 2:</u> M0 for full expansion from the start with no successful factorisation progress M1 A1 M0 M1 A0 for full expansion after one factor found and remaining quadratic factor left unfactorised (or incorrectly done) 4 + M0 for two correct linear factors but final det incorrectly expanded. 5 + A0 for minor sign error, but correct otherwise			
			6	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
3(a)	$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \begin{vmatrix} 1 & 7 & -1 \\ 5 & 1 & 1 \\ 2 & -3 & 1 \end{vmatrix}$	M1		
	$= 1 + 14 + 15 + 2 + 3 - 35 = 0$	A1	2	
	Or			
	$\mathbf{b}\times\mathbf{c} = 4\mathbf{i} - 3\mathbf{j} - 17\mathbf{k}$ and $\begin{bmatrix} 4 \\ -3 \\ -17 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix} = 0$	(M1) (A1)	(2)	Or equivalent
	Or			
	$\mathbf{b} = \mathbf{a} + 2\mathbf{c} \Rightarrow$ co-planarity	(M1) (A1)	(2)	
(b)(i)	$\mathbf{b} - \mathbf{a} = 4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$ $\quad \mathbf{c} - \mathbf{a} = \mathbf{i} - 10\mathbf{j} + 2\mathbf{k}$	B1		Either correct
	$(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -6 & 2 \\ 1 & -10 & 2 \end{vmatrix}$	M1		Genuine attempt using their two vectors
	$= 8\mathbf{i} - 6\mathbf{j} - 34\mathbf{k}$	A1	3	CSO
(ii)	Area $\Delta ABC = \frac{1}{2}$ | this vector |	M1		Must be “Hence” method
	$= \frac{1}{2} \times 2\sqrt{4^2 + 3^2 + 17^2}$	M1		Correct modulus attempt
	$= \sqrt{314}$ or 17.7(2)	A1$\sqrt{}$	3	ft (b)(i) only
			8	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
4(a)	$\Delta = \begin{vmatrix} k & 2 & 1 \\ 1 & k+1 & -2 \\ 2 & -k & 3 \end{vmatrix}$			
	$= 3k^2 + 3k - k - 8 - 2(k+1) - 2k^2 - 6$	M1		Genuine attempt at Δ
	$= k^2 - 16$	A1		
	When $k^2 = 16$ $\Delta = 0 \Rightarrow$ no unique soln.	E1	3	Explained
	Or Subst^g. **Both** $k = 4$ and $k = -4$ and attempt at det.	(M1)		
	Each case correctly shown	(A1)		
		(A1)	3	
(b)	$k = 4 \Rightarrow$ $4x + 2y + z = 5$; $x + 5y - 2z = 3$; $2x - 4y + 3z = -11$	B1		
	Elim^g. z from **(1) & (2)** $\Rightarrow$ $9(x + y) = 13$	M1		Eliminating one variable
	(1) & (3) $\Rightarrow$ $10(x + y) = 26$	A1		Twice, correctly
	Or **(2) & (3)** $\Rightarrow$ $7(x + y) = -13$			
	Explaining inconsistency,eg from $\frac{13}{9} \neq \frac{26}{10}$	E1	4	
	Alternatively (mark as above)			
	Elim^g. x from **(1) & (2)** $\Rightarrow$ $9(2y - z) = 7$			
	(2) & (3) $\Rightarrow$ $7(2y - z) = 17$			
	(1) & (3) $\Rightarrow$ $5(2y - z) = 27$			
	Or			
	Elim^g. y from **(1) & (2)** $\Rightarrow$ $9(2x + z) = 19$			
	(2) & (3) $\Rightarrow$ $7(2x + z) = -43$			
	(1) & (3) $\Rightarrow$ $5(2x + z) = -1$			
(c)(i)	$k = -4 \Rightarrow$ $-4x + 2y + z = 5$; $x - 3y - 2z = 3$; $2x + 4y + 3z = -11$	B1		
	Eliminating one variable	M1		Any pair of equations
	$-7x + y = 13$			
	Or $10y + 7z = -17$			
	Or $10x + z = -21$	A1		Correct
	Parametrisation	M1		Or equivalent
	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 13 \\ -21 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 7 \\ -10 \end{pmatrix}$	A1	5	Any correct answer in any form
	Correct alternate answer forms:			
	x, $y = 13 + 7x$, $z = -21 - 10x$			
	y, $x = (y - 13) / 7$, $z = (-21 - 10y) / 7$			
	z, $y = (-17 - 7z) / 10$, $x = (-21 - z) / 10$			Or equivalents
	Do not accept a mixed parametrisation			
(ii)	The line of intersection of 3 planes	B1	1	Or "Sheaf" of planes
			13	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
5(a)	$\lambda = -4$ gives $P(-29, 42, -19)$ on l	B1	1	Correct value of λ
(b)(i)	$\sqrt{8^2+4^2+1^2} = 9$	B1		Can be awarded retrospectively in (b)(ii) if (b)(i) not done
	dir. cos.s are $\frac{8}{9}, -\frac{4}{9}, \frac{1}{9}$	B1√	2	ft denom[r].
(ii)	$\cos^{-1}\frac{1}{9}$ or 83.6° (or 84°) or 1.46 rads.	B1√	1	ft from 3[rd] d.c. or by any other method (e.g. scalar product) N.B. Mark lost if 6.4° is then offered as the answer
(c)(i)	$\mathbf{n} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$	B1	1	
(ii)	Use of $\cos\theta = \frac{\text{scalar product}}{\text{product of moduli}}$	M1		Must be direction vector of l and their **n**
	Nr. = 45 Dr. = $\sqrt{50}$ .9	A1 A1		ft the "9" if necessary from (b) (i)
	$\theta = 45°$	A1	4	CAO
(d)	Subst[g]. $\begin{pmatrix} 3+8\lambda \\ 26-4\lambda \\ \lambda-15 \end{pmatrix}$ in $3x - 4y + 5z = 100$	M1		$3(3 + 8\lambda) - 4(26 - 4\lambda) + 5(\lambda - 15) = 100$
	Solving a linear eqn. in λ	dM1		
	$\lambda = 6$	A1		CAO
	$\Rightarrow Q = (51, 2, -9)$	B1√	4	ft their λ in l
(e)	$PQ = \sqrt{80^2+40^2+10^2} = 90$	B1		ft
	Sh. Dist.= 90 sin 45° = $45\sqrt{2}$ or 63.6(4…)	M1 A1√	3	ft
	Or $\mathbf{p} + m\,\mathbf{n}$ subst[d]. into $\Pi \Rightarrow m = 9$	(M1)		
	$\Rightarrow R = (-2, 6, 26)$	(A1)		R = foot of perp[r]. from P to Π
	$PR = \sqrt{27^2+36^2+45^2} = 45\sqrt{2}$	(B1√)	(3)	ft
			16	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
6(a)(i)	**AB** = a 3×3 matrix	M1		
	$=\begin{pmatrix}3 & 2 & t+1\\ 1 & 2 & t-1\\ 3 & 2 & t+1\end{pmatrix}$	A1		At least 5 elements correct, incl. at least one from C_3
		A1	3	All elements correct
(ii)	**BA** = a 2×2 matrix	M1		
	$=\begin{pmatrix}2 & 2\\ t & t+4\end{pmatrix}$	A1	2	
(b)	$R_1 = R_3$ ($\Rightarrow$ det **AB** = 0)	B1	1	Or expanding and **showing** det = 0
(c)	**BA** = $\begin{pmatrix}2 & 2\\ -2 & 2\end{pmatrix} = 2\sqrt{2}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$	M1 A1		
	E: enlargement s.f. $2\sqrt{2}$	B1		
	F: Rotation	M1		NB: Rotation bit may be sorted completely separately in which case marks are split 3 + 3
	clockwise (about O) thro' 45°	A1 A1	6	Or -45°, 315°
			12	
7(a)(i)	det **M** = 1 $\Rightarrow$ **area** invariant	B1B1	2	
(ii)	λ^2 – (trace **M**)λ + (det **M**) = 0	M1		
		A1	2	Answer given; condone lack of "= 0"
(iii)	$\lambda = 1$ subst[d]. back $\Rightarrow$ $-2x + 2y = 0$	M1 A1		
	and evec. is $\alpha\begin{pmatrix}1\\1\end{pmatrix}$	A1	3	Any non-zero multiple will do
(iv)	$y = x$ (since $\lambda = 1$) or vector eqn.	B1	1	CAO unless following obviously incorrect working
(b)(i)	$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$	B1 B1	2	Including "= 0" here to be an eqn.
(ii)	det **S** = 1	B1		
	$\Rightarrow$ $ad - bc = 1$	B1√	2	ft 2[nd] B1 from numerical det **S**
(iii)	λ=1 twice gives Char. Eqn. $\lambda^2 - 2\lambda + 1 = 0$	M1		
	$\Rightarrow$ $a + d = 2$	A1	2	CSO
	Or Subst[g]. $\lambda = 1$ in Char. Eqn.			
	$\Rightarrow$ $1 - (a+d) + (ad - bc) = 0$	(M1)		
	and $ad - bc = 1$ $\Rightarrow$ $a + d = 2$	(A1)	(2)	CSO
	Total		**14**	
	TOTAL		**75**	

AQA – Further pure 4 – Jun 2007 – Answers

Question 1:

$a)\ \mathbf{c}\times\mathbf{a} = -\mathbf{a}\times\mathbf{c} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$

$b)\ \mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c} = 2\mathbf{i}-\mathbf{j}+2\mathbf{k}$

$$c)\ (\mathbf{a}\times\mathbf{b}).(\mathbf{a}\times\mathbf{c}) = \begin{pmatrix}3\\1\\1\end{pmatrix}.\begin{pmatrix}-1\\-2\\1\end{pmatrix} = -3-2+1 = -4$$

$d)\ \mathbf{a}.(\mathbf{a}\times\mathbf{c}) = 0$ because $\mathbf{a}$ and $\mathbf{a}\times\mathbf{c}$ are perpendicular.

Exam report

This was a straightforward opener, and was usually handled very well indeed. There was no need to provide reasons to support answers here, though it was pleasing to see many give suitable explanations. In the case of part (d), however, it would have been better for some candidates not to have attempted an explanation, and they shot themselves in the foot in so doing. For instance, the simple statement **a .** (**a** × **c**) = 0 would have scored the mark. But many candidates invented a new distributive law and wrote **a .** (**a** × **c**) = **a . a** × **a . c** = 0 and lost the mark for getting the right answer but for the wrong reason.

Question 2:

$$\begin{vmatrix} y & x & x+y-1 \\ x & y & 1 \\ y+1 & x+1 & 2 \end{vmatrix} = \begin{vmatrix} y-x & x & x+y-1 \\ x-y & y & 1 \\ y-x & x+1 & 2 \end{vmatrix}$$

$$= (y-x)\begin{vmatrix} 1 & x & x+y-1 \\ -1 & y & 1 \\ 1 & x+1 & 2 \end{vmatrix} = (y-x)\begin{vmatrix} 0 & x+y & x+y \\ -1 & y & 1 \\ 1 & x+1 & 2 \end{vmatrix}$$

$$= (y-x)(x+y)\begin{vmatrix} 0 & 1 & 1 \\ -1 & y & 1 \\ 1 & x+1 & 2 \end{vmatrix} = (y-x)(x+y)\begin{vmatrix} 0 & 1 & 1 \\ 0 & x+y+1 & 3 \\ \boxed{1} & x+1 & 2 \end{vmatrix}$$

$$= (y-x)(x+y)(2-x-y)$$

Exam report

The manipulation of determinants continues to be a problem area for many candidates. Expanding from the outset is not really a good idea at all, and all who did so failed to cope satisfactorily with the resulting cubic expression. Even amongst those who found one linear factor before expanding, it was almost invariably the case that they were unable to cope with the resulting quadratic factor. This is very disappointing in its own right, especially when they often had a difference of two squares expression clearly written out in front of them.
Many candidates who used the expected row/column operations approach seemed ill-inclined to state anywhere what operations they were doing. This leaves the markers the task of guessing or deciphering the intended approach, and marks are not credited if the working proves too obscure to figure out.

Question 3:

$$a)\ \mathbf{a}.(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 7 & -1 \\ 5 & 1 & 1 \\ 2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 7 & -1 \\ 6 & 8 & 0 \\ 3 & 4 & 0 \end{vmatrix} = -1\begin{vmatrix} 6 & 8 \\ 3 & 4 \end{vmatrix} = 0$$

$\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ are coplanar

$$b)\ i)\ (\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a}) = \begin{pmatrix}4\\-6\\2\end{pmatrix}\times\begin{pmatrix}1\\-10\\2\end{pmatrix} = \begin{pmatrix}-12+20\\-8+2\\-40+6\end{pmatrix} = \begin{pmatrix}8\\-6\\-34\end{pmatrix}$$

$ii)$ The area of the triangle $ABC = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\left|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\right|$

$$Area = \frac{1}{2}\sqrt{8^2+(-6)^2+(-34)^2} = \sqrt{314} = 17.7 \text{ to 3 sig. fig.}$$

Exam report

This was one of the most successfully attempted questions on the paper for most candidates, although marks were often lost through a lack of care with signs somewhere along the line. In part (a), a small number of candidates evaluated the determinant for the scalar triple product using a calculator (it was presumed) in order to show that it is zero. This means that they showed no working, their solution thus being indistinguishable from the working of a candidate who is unable to evaluate it and simply states it is zero. No marks were awarded in such cases. (For those who seemingly repeated this process in question 4(a), marks were given **bod** in question 4 on the basis that we were not happy with penalising them a further three marks just for having a useful calculator facility). Many forgot the factor of ½ in part (b) (ii), but otherwise this was a high-scoring question for the majority of candidates.

Question 4:	Exam report

Let's work out the determinant of the associated matrix:

$$\begin{vmatrix} k & 2 & 1 \\ 1 & k+1 & -2 \\ 2 & -k & 3 \end{vmatrix} = k\begin{vmatrix} k+1 & -2 \\ -k & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & k+1 \\ 2 & -k \end{vmatrix}$$

$$= k(3k+3-2k) - 2(3+4) + (-k-2k-2)$$

$$= k^2 + 3k - 14 - 3k - 2 = k^2 - 16.$$

The system does not have a unique solution when the determinant is 0. This happens for $k^2 = 16$.

$b)$ For $k = 4$, the system becomes

$$\begin{cases} 4x+2y+z=5 \\ x+5y-2z=3 \\ 2x-4y+3z=-11 \end{cases} \Leftrightarrow \begin{cases} x+5y-2z=3 \quad l_2 \\ 18y-9z=7 \quad (4l_2-l_1) \\ 14y-7z=17\,(2l_2-l_3) \end{cases} \Leftrightarrow \begin{cases} x+5y-2z=3 \\ 2y-z=\frac{7}{9} \\ 2y-z=\frac{17}{7} \end{cases}$$

The system is inconsistent.

For $k = -4$, the system becomes

$$\begin{cases} -4x+2y+z=5 \\ x-3y-2z=3 \\ 2x+4y+3z=-11 \end{cases} \Leftrightarrow \begin{cases} x-3y-2z=3 \\ -10y-7z=17 \\ -10y-7z=17 \end{cases} \Leftrightarrow \begin{cases} x=3+3y+2z=-\frac{21}{10}-\frac{1}{10}t \\ y=-\frac{17}{10}-\frac{7}{10}t \\ z=t \end{cases}$$

This is a line going through the point $(-2.1, -1.7, 0)$ with direction vector $-\frac{1}{10}\mathbf{i}-\frac{7}{10}\mathbf{j}+\mathbf{k}$

An equation of this line is : $\mathbf{r} = \begin{pmatrix} -2.1 \\ -1.7 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 7 \\ -10 \end{pmatrix}$

$ii)$ The three planes intersect at this line.

Exam report

This algebra question was the one that produced the most variable set of responses from candidates, from the outstanding to the careless to the .haven.t got a clue what to do. variety. In part (a), the general approach is to find the determinant of the coefficient matrix in terms of *k* and see when this is zero. The alternative is to evaluate it using *k* = 4 **and** .-4. It was really shocking to see how few candidates appeared to realise that k^2 = 16 gave rise to the two values of *k*. As mentioned earlier, the manipulation of some fairly simple equations in order to eliminate one or other of the three variables left a lot to be desired, and many candidates would have scored several more marks with just a little more care.

Moreover, the question itself implies that there are no solutions to part (b) and infinitely many in part (c), so it was surprising to see so many submissions moving towards a unique solution to the system in these cases. Finally, in part (b), the widespread inability of candidates to demonstrate an inconsistency was disappointing. Very few candidates did so successfully, predominantly because much of their prior working was error-strewn, rendering a valid conclusion impossible, despite their claims.

Question 5:	Exam report
$a)\, P(-29, 42, -19)$ Is there a value of λ for which $\begin{cases} 3+8\lambda = -29 \\ 26-4\lambda = 42 \\ -15+\lambda = -19 \end{cases}$? $\begin{cases} \lambda = -4 \\ \lambda = -4 \\ \lambda = -4 \end{cases}$ P belongs to the line l. $b)$ A direction vector of the line is $\mathbf{u}=\begin{pmatrix} 8 \\ -4 \\ 1 \end{pmatrix}$ $\lvert\mathbf{u}\rvert = \sqrt{8^2+(-4)^2+1^2} = \sqrt{81} = 9$ The direction cosines are : $\frac{8}{9}, \frac{-4}{9}, \frac{1}{9}$ $ii)$ The cosine of the angle between the line and the z-axis is $\frac{1}{9}$ $Cos^{-1}\frac{1}{9} = 83.6^o = 1.46\ rad$ $c)\, i)$ A normal vector to the plane Π is $\mathbf{n}=\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$ $ii)$ Let's work out the angle between $\mathbf{u}=\begin{pmatrix} 8 \\ -4 \\ 1 \end{pmatrix}$ and $\mathbf{n}=\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$ $\mathbf{u.n} = \begin{pmatrix} 8 \\ -4 \\ 1 \end{pmatrix}.\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} = 24+16+5 = 45$, $\lvert\mathbf{u}\rvert = 9$ and $\lvert\mathbf{n}\rvert = \sqrt{9+16+25} = \sqrt{50} = 5\sqrt{2}$ $Cos\alpha = \frac{\mathbf{u.n}}{\lvert\mathbf{u}\rvert\lvert\mathbf{n}\rvert} = \frac{45}{9\times5\sqrt{2}} = \frac{1}{\sqrt{2}}$ $\alpha = 45^o$ $d)$ Q belongs to the line so we can write $Q(3+8\lambda, 26-4\lambda, -15+\lambda)$ and Q belongs to the plane so $3x_Q - 4y_Q + 5z_Q = 100$ $3(3+8\lambda)-4(26-4\lambda)+5(-15+\lambda) = 100$ $9+24\lambda-104+16\lambda-75+5\lambda = 100$ $45\lambda - 170 = 100$ $\lambda = 6$ The coordinates of Q are $(51, 2, -9)$ and its position vector is $\mathbf{q}=\begin{pmatrix} 51 \\ 2 \\ -9 \end{pmatrix}$ $e)$ The distance PQ=$\lvert\overline{PQ}\rvert = \lvert\mathbf{q}-\mathbf{p}\rvert$ $\mathbf{q}-\mathbf{p} = 80\mathbf{i}-40\mathbf{j}+10\mathbf{k}$ and $\lvert\mathbf{q}-\mathbf{p}\rvert = \sqrt{80^2+(-40)^2+10^2} = 90$ The shortest distance from P to the plane Π is PQSinα=$90\times\sin45^o = 45\sqrt{2}$	Parts (a) to (d) of this question were usually very well done, and candidates were able to score a lot of the marks. The big surprise came in part (e) when so few arrived at the correct final answer. In almost all cases, a simple diagram would have helped enormously, and they would then have seen that a simple bit of basic trigonometry would have done the trick. In most cases, candidates seemed to be working with randomly-selected vectors from the first few parts of the question and trying to insert them into some (often half-) remembered vector formula.

Question 6:	Exam report
$a)\, i)\ \mathbf{AB} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & t \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1+t \\ 1 & 2 & -1+t \\ 3 & 2 & 1+t \end{pmatrix}$ $ii)\ \mathbf{BA} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & t \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ t & 4+t \end{pmatrix}$ $b)\ \det(\mathbf{AB}) = \begin{vmatrix} 3 & 2 & 1+t \\ 1 & 2 & -1+t \\ 3 & 2 & 1+t \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 2 & -1+t \\ 3 & 2 & 1+t \end{vmatrix} = 0$ by replacing R_1 *by* $R_1 - R_3$ **AB** is singular for all values of t. $c)$ For $t = -2$, $\mathbf{BA} = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}$ $\sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$ $\mathbf{BA} = 2\sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ E is an enlargement about O(0,0) scale factor $2\sqrt{2}$ F is a rotation about O(0,0), angle 45^o *clockwise.*	The two product matrices **AB** and **BA** in part (a) of this question were very popular and the majority of candidates gained all 5 marks. The rest of the question really was poorly done on the whole, but this was the most demanding work on the paper. In part (b), many of candidates noted that det (**AB**) was zero, but failed to explain why. In part (c), most seemed to resort to guesswork, especially regarding the transformation F. A shear was, marginally, the most popular choice, with a 90^o rotation close behind. The majority of candidates seemed to have no method of approach to the problem. Those that took a scalar factor out of the matrix usually contented themselves with 2, 4 or 8 as the scale factor of the enlargement E; thereby leaving themselves with a matrix they simply did not know what to do with. Those who chose .rotation. as their answer then did so from a matrix with det ≠ 1 and consequently did not gain all the marks available.

Question 7:	Exam report
$a)\ i)\ Det(\mathbf{M}) = \begin{vmatrix} -1 & 2 \\ -2 & 3 \end{vmatrix} = -3+4 = 1$ The shear transform a shape into a shape with same area. (The area is invariant). $ii)\ \det(\mathbf{M}-\lambda\mathbf{I}) = \begin{vmatrix} -1-\lambda & 2 \\ -2 & 3-\lambda \end{vmatrix} = (-1-\lambda)(3-\lambda)+4$ $= \lambda^2 - 2\lambda + 1$ The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$ $iii)\ \lambda^2 - 2\lambda + 1 = 0$ $(\lambda-1)^2 = 0 \qquad \lambda = 1\ (repeated\ \ value)$ To find an eigen vector, let's solve $(\mathbf{M}\text{-}\lambda\mathbf{I})\begin{pmatrix} x \\ y \end{pmatrix} = 0$ $\Leftrightarrow \begin{cases} -2x+2y=0 \\ -2x+2y=0 \end{cases} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector. $iv)$ The line of invariant points has direction vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and goes through (0,0), its equation is $y = x$. $b)\ i)\ \det(\mathbf{S}-\lambda\mathbf{I}) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda)-bc$ $\det(\mathbf{S}-\lambda\mathbf{I}) = \lambda^2 - (a+d)\lambda + (ad-bc) = 0$ is the characteristic equation. $ii)$ Through a shear, the area is invariant, so det(**S**)=1 $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1 \Leftrightarrow ad - bc = 1$ $iii)$ 1 is a repeated eigenvalue so the characteristic equation is $(\lambda-1)^2 = 0 \Leftrightarrow \lambda^2 - 2\lambda + 1 = 0 \qquad so\ a+d=2$	This question was so structured that, carelessness apart, it proved a very good source of marks and the majority of candidates found it so, if not in its entirety then at least in part.

Grade boundaries							
Grade			A	B	C	D	E
Mark	Max 75		62	54	46	38	31