General Certificate of Education
Advanced Level Examination
January 2012

# Mathematics MFP4

## Unit Further Pure 4

## Friday 27 January 2012 9.00 am to 10.30 am

**For this paper you must have:**
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

**Time allowed**
- 1 hour 30 minutes

**Instructions**
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

**Information**
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

**Advice**
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

**1** The vectors $\mathbf{a}$ and $\mathbf{b}$ are such that $\mathbf{a}.\mathbf{b} = 21$, $|\mathbf{a}| = 5\sqrt{2}$ and $|\mathbf{b}| = 3$.

Determine the exact value of $|\mathbf{a} \times \mathbf{b}|$. *(5 marks)*

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**2** Describe the single transformation represented by each of the matrices:

**(a)** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; *(2 marks)*

**(b)** $\begin{bmatrix} 0.6 & 0 & -0.8 \\ 0 & 1 & 0 \\ 0.8 & 0 & 0.6 \end{bmatrix}$. *(3 marks)*

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**3 (a)** Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{M} = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}$. *(6 marks)*

**(b)** The plane transformation T is given by the matrix $\mathbf{M}$. Write down the coordinates of the invariant point of T. *(1 mark)*

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**4** Let $\mathbf{X} = \begin{bmatrix} 3 & x \\ -1 & 7 \end{bmatrix}$.

**(a)** Determine $\mathbf{X}\mathbf{X}^{\mathrm{T}}$. *(2 marks)*

**(b)** Show that $\mathrm{Det}(\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}^{\mathrm{T}}\mathbf{X}) \leqslant 0$ for all real values of $x$. *(4 marks)*

**(c)** Find the value of $x$ for which the matrix $(\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}^{\mathrm{T}}\mathbf{X})$ is singular. *(1 mark)*

**5 (a)** Determine the two values of the integer $n$ for which the system of equations

$$\begin{aligned} 2x + ny + \phantom{7}z &= 5 \\ 3x - \phantom{7}y + nz &= 1 \\ -x + 7y + \phantom{7}z &= n \end{aligned}$$

does not have a unique solution. *(4 marks)*

**(b)** For the positive value of $n$ found in part **(a)**, determine whether the system is consistent or inconsistent, and interpret this result geometrically. *(6 marks)*

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**6** The planes $\Pi_1$ and $\Pi_2$ have equations

$$\mathbf{r} \cdot \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix} = 10 \quad \text{and} \quad \mathbf{r} \cdot \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = 7$$

respectively.

**(a)** Determine, to the nearest degree, the acute angle between $\Pi_1$ and $\Pi_2$. *(4 marks)*

**(b)** By setting $z = t$, find cartesian equations for the line of intersection of $\Pi_1$ and $\Pi_2$ in the form

$$\frac{x-a}{l} = \frac{y-b}{m} = z = t$$ *(6 marks)*

**(c)** The line $L$, with equation $\mathbf{r} = \begin{bmatrix} 20 \\ -1 \\ 7 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 9 \\ 4 \end{bmatrix}$, intersects $\Pi_1$ at the point $P$ and $\Pi_2$ at the point $Q$.

Show that $PQ = k\sqrt{2}$, where $k$ is an integer. *(6 marks)*

**Turn over ▶**

**7** The plane transformation T is a rotation through $\theta$ radians anticlockwise about $O$, and maps points $(x, y)$ onto image points $(X, Y)$ such that

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $c = \cos\theta$ and $s = \sin\theta$ .

**(a)** Write down the inverse of the matrix $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ and hence show that

$$x = cX + sY \quad \text{and} \quad y = -sX + cY$$ *(3 marks)*

**(b)** The curve $C$ has equation $x^2 - 6xy - 7y^2 = 8$ .

The image of $C$ under T is the curve $C'$ with equation $pX^2 + qXY + rY^2 = 8$ .

**(i)** Use the results of part **(a)** to show that

$$q = 6s^2 + 16sc - 6c^2$$

and express $p$ and $r$ similarly in terms of $c$ and $s$. *(4 marks)*

**(ii)** Given that $\theta$ is an acute angle, find the values of $c$ and $s$ for which $q = 0$ and hence in this case express the equation of $C'$ in the form

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$$ *(8 marks)*

**(iii)** Hence explain why $C$ is a hyperbola. *(1 mark)*

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**8** For $n \neq 1$, the vectors **a**, **b** and **c** are such that

$$\mathbf{a} = \begin{bmatrix} 1 \\ n \\ n^2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2n \\ 2n^2 + n \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} n - 1 \\ n^2 - 1 \\ 1 - n^2 \end{bmatrix}$$

Determine the value of $n$ for which **a**, **b** and **c** are linearly dependent. *(9 marks)*

**Key to mark scheme abbreviations**

| | |
|---|---|
| M | mark is for method |
| m or dM | mark is dependent on one or more M marks and is for method |
| A | mark is dependent on M or m marks and is for accuracy |
| B | mark is independent of M or m marks and is for method and accuracy |
| E | mark is for explanation |
| √or ft or F | follow through from previous incorrect result |
| CAO | correct answer only |
| CSO | correct solution only |
| AWFW | anything which falls within |
| AWRT | anything which rounds to |
| ACF | any correct form |
| AG | answer given |
| SC | special case |
| OE | or equivalent |
| A2,1 | 2 or 1 (or 0) accuracy marks |
| $-x$ EE | deduct $x$ marks for each error |
| NMS | no method shown |
| PI | possibly implied |
| SCA | substantially correct approach |
| c | candidate |
| sf | significant figure(s) |
| dp | decimal place(s) |

**No Method Shown**

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

**Otherwise we require evidence of a correct method for any marks to be awarded.**

**MFP4**

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 1 | Use of $ab\cos\theta$ = **a . b** = 21<br>$\Rightarrow\ \cos\theta = \frac{7}{5\sqrt{2}}$ | M1 A1 | | |
| | $\Rightarrow\ \sin\theta = \frac{1}{5\sqrt{2}}$ | B1 ft | | FT exact only |
| | Use of $\lvert \mathbf{a} \times \mathbf{b} \rvert = ab\sin\theta = 3$ | M1 A1 | 5 | CSO |
| | **Total** | | **5** | |
| 2(a) | Reflection in $x = z$ | M1 A1 | 2 | |
| (b) | Rotation about the $y$-axis | M1 A1 | | |
| | Through $\cos^{-1}0.6$ ($\approx 53.13^{\circ}$) | A1 | 3 | Ignore direction |
| | **Total** | | **5** | |
| 3(a) | Char. Eqn. is $\lambda^2 - 8\lambda - 9 = 0$ | M1 | | Attempted |
| | Quadratic solved to get two roots | dM1 | | |
| | $\Rightarrow\ \lambda = 9, -1$ | A1 | | |
| | Subst$^{g}$. back $\lambda$ (at least once) : | M1 | | |
| | $\lambda = 9\ \Rightarrow\ -x + y = 0$<br>$\Rightarrow\ \lambda = 9$ has evecs. $\alpha\begin{bmatrix}1\\1\end{bmatrix}$ | A1 | | any $\alpha \neq 0$ |
| | $\lambda = -1\ \Rightarrow\ x + y = 0$<br>$\Rightarrow\ \lambda = -1$ has evecs. $\beta\begin{bmatrix}1\\-1\end{bmatrix}$ | A1 | 6 | any $\beta \neq 0$ |
| (b) | (0, 0) | B1 | 1 | |
| | **Total** | | **7** | |

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 4(a) | $\mathbf{X\,X^T} = \begin{bmatrix} 3 & x \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ x & 7 \end{bmatrix}$ | M1 | | Attempted multn. with $\mathbf{X^T}$ correct |
| | $= \begin{bmatrix} x^2+9 & 7x-3 \\ 7x-3 & 50 \end{bmatrix}$ | A1 | 2 | |
| (b) | $\mathbf{X^TX} = \begin{bmatrix} 3 & -1 \\ x & 7 \end{bmatrix}\begin{bmatrix} 3 & x \\ -1 & 7 \end{bmatrix}$ | | | |
| | $= \begin{bmatrix} 10 & 3x-7 \\ 3x-7 & x^2+49 \end{bmatrix}$ | M1 | | Good attempt |
| | $\mathbf{X\,X^T} - \mathbf{X^TX} = \begin{bmatrix} x^2-1 & 4x+4 \\ 4x+4 & 1-x^2 \end{bmatrix}$ | M1 | | Good attempt |
| | Det($\mathbf{X\,X^T} - \mathbf{X^TX}$) = $\begin{vmatrix} x^2-1 & 4x+4 \\ 4x+4 & 1-x^2 \end{vmatrix}$ | | | |
| | $= (x+1)^2 \begin{vmatrix} x-1 & 4 \\ 4 & 1-x \end{vmatrix}$ | M1 | | Good attempt to factorise/expand the determinant |
| | $= -(x+1)^2\{(x-1)^2+16\} \le 0$ for all real $x$ | E1 | 4 | Explained/demonstrated fully |
| (c) | $x = -1$ | B1 | 1 | CSO |
| | **Total** | | **7** | |

**MFP4 (cont)**

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| **5(a)** | $\begin{vmatrix} 2 & n & 1 \\ 3 & -1 & n \\ -1 & 7 & 1 \end{vmatrix}$ | | | |
| | Expanding the det. of the coefft. mtx.<br>Setting it = 0<br>Obtaining & solving a quadratic eqn. in $n$ | M1<br>M1<br>M1 | | |
| | $0 = n^2 + 17n - 18 = (n + 18)(n - 1)$<br>$\Rightarrow\ n = 1, -18$ | A1 | 4 | CSO |
| **(b)** | $n = 1$ gives $2x + y + z = 5$<br>$3x - y + z = 1$<br>$-x + 7y + z = 1$ | B1 | | ft their chosen integer $n$ |
| | Eliminating one variable from a pair of equations, twice | M1 | | |
| | e.g. ② – ① $\Rightarrow$ $x - 2y = -4$<br>and ② – ③ $\Rightarrow$ $4x - 8y = 0$ | A1 ft<br>A1 ft | | |
| | Inconsistency clearly demonstrated from fully correct working | E1 | | |
| | 3 planes have no common intersection (or form a $\Delta^r$ prism) | B1 ft | **6** | Also ft "3 planes meet in a common line" or "3 planes form a sheaf" if consistency conclusion made |
| | **Total** | | **10** | |

**MFP4 (cont)**

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| 6(a) | Use of scalar product on $\begin{pmatrix}2\\1\\7\end{pmatrix}$ and $\begin{pmatrix}3\\1\\-4\end{pmatrix}$ | M1 | | |
| | Sc.Prod. = ± 21 | B1 | | |
| | Moduli $\sqrt{54}$ and $\sqrt{26}$ correct | B1 | | Accept 7.348… & 5.099… |
| | AWRT 56° | A1 | 4 | From correct working |
| (b) | $2x + y + 7t = 10$ and $3x + y - 4t = 7$ noted or used | M1 | | |
| | Eliminating (say) $y$ to get $x$ as a fn. of $t$ | M1 | | |
| | $x = 11t - 3$ | A1 | | CAO |
| | Subst$^g$. back for $y$ | M1 | | |
| | $y = 16 - 29t$ | A1 | | CAO |
| | $\dfrac{x+3}{11} = \dfrac{y-16}{-29} = z \quad (= t)$ | B1 ft | 6 | |
| (c) | Attempt at either $\begin{pmatrix}\lambda+20\\9\lambda-1\\4\lambda+7\end{pmatrix} \bullet \begin{pmatrix}2\\1\\7\end{pmatrix}$ = 10 or $\begin{pmatrix}\lambda+20\\9\lambda-1\\4\lambda+7\end{pmatrix} \bullet \begin{pmatrix}3\\1\\-4\end{pmatrix}$ = 7 | M1 | | |
| | Solving either<br>$40 + 2\lambda - 1 + 9\lambda + 49 + 28\lambda = 10$ or<br>$60 + 3\lambda - 1 + 9\lambda - 28 - 16\lambda = 7$ | M1 | | |
| | $\lambda_1 = -2$ $\quad$ $\lambda_2 = 6$ | A1A1 | | |
| | $P = (18, -19, -1)$<br>and $Q = (26, 53, 31)$ | | | |
| | $PQ =$<br>$\sqrt{8^2 + 72^2 + 32^2} = \sqrt{6272} = 56\sqrt{2}$ | M1A1 | 6 | **NB** $P$, $Q$ not required:<br>$d = \lvert \lambda_1 - \lambda_2 \rvert \times \lvert \mathbf{i} + 9\mathbf{j} + 4\mathbf{k} \rvert$<br>$= 8 \times 7\sqrt{2} = 56\sqrt{2}$ **M1 A1** |
| | **Total** | | **16** | |

**MFP4 (cont)**

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| **7(a)** | $\begin{bmatrix} c & s \\ -s & c \end{bmatrix}$ | B1 | | |
| | $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}$ | M1 | | |
| | $= \begin{bmatrix} cX + sY \\ -sX + cY \end{bmatrix}$ | A1 | 3 | |
| **(b)(i)** | $(cX+sY)^2 - 6(cX+sY)(-sX+cY) - 7(-sX+cY)^2 = 8$ | | | |
| | $(c^2X^2 + 2csXY + s^2Y^2) - 6([c^2 - s^2]XY + sc[Y^2 - X^2])$ $-7(s^2X^2 - 2csXY + c^2Y^2) = 8$ | M1 | | Substn. for *x* & *y* in eqn. *and* multiplying out |
| | $p = c^2 + 6sc - 7s^2$ | A1 | | |
| | $q = 16cs - 6(c^2 - s^2)$ | A1 | | AG |
| | $r = s^2 - 6sc - 7c^2$ | A1 | 4 | |
| **(ii)** | Factorising: $3s^2 + 8sc - 3c^2 = (3s - c)(s + 3c) = 0$ | M1A1 | | Or by double angles |
| | Deducing a tan value | M1 | | |
| | $\tan\theta = \frac{1}{3}$ ($\theta$ acute) | A1 | | |
| | $\cos\theta = \dfrac{3}{\sqrt{10}}$, $\sin\theta = \dfrac{1}{\sqrt{10}}$ | A1 | | Both |
| | Subst$^{g}$. sensible values back for *p* and *r* | M1 | | |
| | $2X^2 - 8Y^2 = 8$ | A1 | | |
| | $\dfrac{X^2}{2^2} - \dfrac{Y^2}{1^2} = 1$ | A1 | 8 | CSO |
| **(iii)** | Since $C'$ is a hyperbola, and it is just $C$ rotated, it follows that $C$ is a hyperbola | E1 | 1 | |
| | **Total** | | **16** | |

**MFP4 (cont)**

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| **8** | For considering $\begin{vmatrix} 1 & 2n & n-1 \\ n & 2n^2+n & n^2-1 \\ n^2 & -1 & 1-n^2 \end{vmatrix}$ | B1 | | Or by scalar triple product |
| | $= (n-1)\begin{vmatrix} 1 & 2n & 1 \\ n & 2n^2+n & n+1 \\ n^2 & -1 & -1-n \end{vmatrix}$ | M1A1 | | For 1st factor |
| | $= (n-1)\begin{vmatrix} 1 & 2n & 1 \\ n & 2n^2+n & n+1 \\ n(n+1) & (n+1)(2n-1) & 0 \end{vmatrix}$ | M1 | | Row ops. for 2nd factor |
| | $R_3' = R_3 + R_2$ | | | |
| | $= (n-1)(n+1)\begin{vmatrix} 1 & 2n & 1 \\ n & 2n^2+n & n+1 \\ n & 2n-1 & 0 \end{vmatrix}$ | M1A1 | | |
| | $= (n-1)(n+1)$ $\{2n^3+2n^2+2n^2-n-2n^3-n^2-2n^2-n+1\}$ | | | |
| | **OR** | | | |
| | $= (n-1)(n+1)\begin{vmatrix} 1 & 2n & 1 \\ 0 & n & 1 \\ n & 2n-1 & 0 \end{vmatrix}$ | | | |
| | $R_2' = R_2 - nR_1$ = | | | |
| | $(n-1)(n+1)\{2n^2-n^2-2n+1\}$ | M1 | | Full method for remaining factors |
| | $= (n-1)(n+1)(n-1)^2$ | A1 | | |
| | $n = -1$ | B1 | 9 | CSO |
| | | | | **Note**: Expanding straightaway scores **B1 M1** and then **A1** for $n^4 - 2n^3 + 2n - 1$ Thereafter, **M1 A1** for 1st factor, **M1** for 2nd factor attempted and **M1** for full method for remaining factors plus **A1** and **B1 cso** at the end, as above. |
| | **Total** | | **9** | |
| | **TOTAL** | | **75** | |

## Scaled mark unit grade boundaries - January 2012 exams

### A-level

| Code | Title | Max. Scaled Mark | A* | A | B | C | D | E |
|---|---|---|---|---|---|---|---|---|
| | | | Scaled Mark Grade Boundaries and A* Conversion Points | | | | | |
| LAW02 | LAW UNIT 2 | 94 | - | 73 | 66 | 59 | 52 | 46 |
| LAW03 | LAW UNIT 3 | 80 | 69 | 63 | 57 | 51 | 45 | 40 |
| | | | | | | | | |
| MD01 | MATHEMATICS UNIT MD01 | 75 | - | 62 | 56 | 50 | 44 | 39 |
| MFP1 | MATHEMATICS UNIT MFP1 | 75 | - | 67 | 60 | 53 | 46 | 39 |
| MM1A | MATHEMATICS UNIT MM1A | 100 | no candidates were entered for this unit | | | | | |
| MM1B | MATHEMATICS UNIT MM1B | 75 | - | 59 | 52 | 46 | 40 | 34 |
| MPC1 | MATHEMATICS UNIT MPC1 | 75 | - | 61 | 55 | 49 | 43 | 37 |
| | | | | | | | | |
| MS1A | MATHEMATICS UNIT MS1A | 100 | - | 74 | 65 | 56 | 47 | 38 |
| *MS/SS1A/W* | *MATHEMATICS UNIT S1A - WRITTEN* | *75* | | *54* | | | | *28* |
| *MS/SS1A/C* | *MATHEMATICS UNIT S1A - COURSEWORK* | *25* | | *20* | | | | *10* |
| | | | | | | | | |
| MS1B | MATHEMATICS UNIT MS1B | 75 | - | 56 | 49 | 42 | 36 | 30 |
| MD02 | MATHEMATICS UNIT MD02 | 75 | 69 | 64 | 57 | 50 | 44 | 38 |
| MFP2 | MATHEMATICS UNIT MFP2 | 75 | 59 | 52 | 45 | 38 | 31 | 25 |
| MM2B | MATHEMATICS UNIT MM2B | 75 | 69 | 63 | 55 | 47 | 39 | 32 |
| MPC2 | MATHEMATICS UNIT MPC2 | 75 | - | 66 | 59 | 52 | 46 | 40 |
| MS2B | MATHEMATICS UNIT MS2B | 75 | 69 | 63 | 55 | 47 | 40 | 33 |
| MFP3 | MATHEMATICS UNIT MFP3 | 75 | 67 | 60 | 52 | 44 | 37 | 30 |
| MPC3 | MATHEMATICS UNIT MPC3 | 75 | 64 | 57 | 50 | 43 | 37 | 31 |
| MFP4 | MATHEMATICS UNIT MFP4 | 75 | 60 | 54 | 48 | 42 | 37 | 32 |
| MPC4 | MATHEMATICS UNIT MPC4 | 75 | 63 | 57 | 51 | 45 | 39 | 33 |
| | | | | | | | | |
| MEST1 | MEDIA STUDIES UNIT 1 | 80 | - | 55 | 47 | 40 | 33 | 26 |
| MEST2 | MEDIA STUDIES UNIT 2 | 80 | - | 63 | 54 | 45 | 36 | 28 |
| MEST3 | MEDIA STUDIES UNIT 3 | 80 | 67 | 57 | 47 | 37 | 27 | 18 |
| MEST4 | MEDIA STUDIES UNIT 4 | 80 | 74 | 68 | 56 | 45 | 34 | 23 |
| | | | | | | | | |
| PHIL1 | PHILOSOPHY UNIT 1 | 90 | - | 55 | 49 | 43 | 37 | 32 |