General Certificate of Education
Advanced Level Examination
January 2011

Mathematics MFP4

Unit Further Pure 4

Friday 28 January 2011 9.00 am to 10.30 am

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

1 Let $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix}$.

(a) Use a row operation to show that $(x+y+z)$ is a factor of Δ. *(2 marks)*

(b) Hence, or otherwise, express Δ as a product of linear factors. *(2 marks)*

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2 The non-zero vectors **a** and **b** have magnitudes a and b respectively.

Let $c = |\mathbf{a} \times \mathbf{b}|$ and $d = |\mathbf{a} \,.\, \mathbf{b}|$.

By considering the definitions of the vector and scalar products, or otherwise, show that

$$c^2 + d^2 = a^2b^2$$ *(3 marks)*

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3 (a) Find the values of t for which the system of equations

$$\begin{aligned} tx + 2y + \quad 3z &= a \\ 2x + 3y - \quad tz &= b \\ 3x + 5y + (t+1)z &= c \end{aligned}$$

does not have a unique solution. *(3 marks)*

(b) For the integer value of t found in part **(a)**, find the relationship between a, b and c such that this system of equations is consistent. *(3 marks)*

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4 The non-singular matrix $\mathbf{X} = \begin{bmatrix} 3 & 1 & -3 \\ 2 & 4 & 3 \\ -4 & 2 & -1 \end{bmatrix}$.

(a) (i) Show that $\mathbf{X}^2 - \mathbf{X} = k\mathbf{I}$ for some integer k. *(3 marks)*

(ii) Hence show that $\mathbf{X}^{-1} = \frac{1}{20}(\mathbf{X} - \mathbf{I})$. *(2 marks)*

(b) The 3×3 matrix $\mathbf{Y}$ has inverse $\mathbf{Y}^{-1} = \begin{bmatrix} 60 & 0 & 0 \\ 0 & 0 & -10 \\ 0 & 20 & 0 \end{bmatrix}$.

Without finding $\mathbf{Y}$, determine the matrix $(\mathbf{XY})^{-1}$. *(3 marks)*

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5 The planes Π_1 and Π_2 have vector equations $\mathbf{r} \cdot \begin{bmatrix} 6 \\ 2 \\ 9 \end{bmatrix} = 5$ and $\mathbf{r} \cdot \begin{bmatrix} 10 \\ -1 \\ -11 \end{bmatrix} = 4$ respectively.

(a) Write down cartesian equations for Π_1 and Π_2. *(1 mark)*

(b) Find a vector equation for the line of intersection of Π_1 and Π_2. *(5 marks)*

(c) The plane Π_3 has cartesian equation $5x + 3y + 11z = 28$.

Use your answer to part **(b)** to find the coordinates of the point of intersection of Π_1, Π_2 and Π_3. *(4 marks)*

(d) Determine a vector equation for the plane which passes through the point $(4, 1, 9)$ and which is perpendicular to both Π_1 and Π_2. *(3 marks)*

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6 The plane Π has equation $\mathbf{r} \cdot \begin{bmatrix} 12 \\ 15 \\ 16 \end{bmatrix} = 11$ and the point Q has coordinates $(1, 1, -1)$.

(a) Show that Q is in Π. *(1 mark)*

(b) (i) Write down cartesian equations for the line l which passes through Q and is perpendicular to Π. *(2 marks)*

(ii) Deduce the direction cosines of l. *(2 marks)*

(c) The points M and N are on l, and each is 50 units from Π.

Find the coordinates of M and N. *(3 marks)*

(d) Given that the point $P(5, 1, -4)$ is in Π, determine the area of triangle PMN. *(3 marks)*

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7 Let $\mathbf{Y} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.

(a) Show that 4 is a repeated eigenvalue of $\mathbf{Y}$, and find the other eigenvalue of $\mathbf{Y}$. *(7 marks)*

(b) For each eigenvalue of $\mathbf{Y}$, find a full set of eigenvectors. *(5 marks)*

(c) The matrix $\mathbf{Y}$ represents the transformation T.

Describe the geometrical significance of the eigenvectors of $\mathbf{Y}$ in relation to T. *(3 marks)*

Turn over ▶

8 The plane transformation T is represented by the matrix $\mathbf{M} = \begin{bmatrix} -3 & 8 \\ -1 & 3 \end{bmatrix}$.

(a) The quadrilateral $ABCD$ has image $A'B'C'D'$ under T.

Evaluate $\det \mathbf{M}$ and describe the geometrical significance of both its sign and its magnitude in relation to $ABCD$ and $A'B'C'D'$. *(3 marks)*

(b) The line $y = px$ is a line of invariant points of T, and the line $y = qx$ is an invariant line of T.

Show that $p = \frac{1}{2}$ and determine the value of q. *(5 marks)*

(c) (i) Find the 2×2 matrix $\mathbf{R}$ which represents a reflection in the line $y = \frac{1}{2}x$. *(2 marks)*

(ii) Given that T is the composition of a shear, with matrix $\mathbf{S}$, followed by a reflection in the line $y = \frac{1}{2}x$, determine the matrix $\mathbf{S}$ and describe the shear as fully as possible.

(5 marks)

Key to mark scheme abbreviations

M	mark is for method
m or dM	mark is dependent on one or more M marks and is for method
A	mark is dependent on M or m marks and is for accuracy
B	mark is independent of M or m marks and is for method and accuracy
E	mark is for explanation
√or ft or F	follow through from previous incorrect result
CAO	correct answer only
CSO	correct solution only
AWFW	anything which falls within
AWRT	anything which rounds to
ACF	any correct form
AG	answer given
SC	special case
OE	or equivalent
A2,1	2 or 1 (or 0) accuracy marks
$-x$ EE	deduct x marks for each error
NMS	no method shown
PI	possibly implied
SCA	substantially correct approach
c	candidate
sf	significant figure(s)
dp	decimal place(s)

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

MFP4

Q	Solution	Marks	Total	Comments
1(a)	$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ x & y & z \\ x+y+z & y+z+x & z+x+y \end{vmatrix}$	M1		e.g. $R_3{}' = R_3 + R_2$
	$= (x+y+z)\begin{vmatrix} 1 & 2 & 3 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$	A1	2	
(b)	Expanding remaining det.	M1		
	$\Delta = (x+y+z)(x-2y+z)$	A1	2	
	Total		**4**	
2	$c = \lvert \mathbf{a} \times \mathbf{b} \rvert = ab\sin\theta$	B1		
	$d = \lvert \mathbf{a} \bullet \mathbf{b} \rvert = ab\lvert\cos\theta\rvert$	B1		Condone lack of | - |
	$c^2 + d^2 = a^2b^2(\cos^2\theta + \sin^2\theta) = a^2b^2$	B1	3	Legitimately shown
	Total		**3**	
3(a)	$\begin{vmatrix} t & 2 & 3 \\ 2 & 3 & -t \\ 3 & 5 & t+1 \end{vmatrix} = 8t^2 - 7t - 1 = 0$	M1 M1		Attempt at det. of coefft. mtx. (or equivalent) Equating to zero and solving a quadratic eqn. in t
	$t = 1,\ -\frac{1}{8}$	A1	3	
(b)	$x + 2y + 3z = a$ $t = 1 \Rightarrow 2x + 3y - z = b$ $3x + 5y + 2z = c$	B1√		FT any integer value found
	E.g. ① + ② – ③ $\Rightarrow a + b = c$	M1 A1	3	
	Total		**6**	
4(a)	**(i)** $\mathbf{X}^2 = \begin{bmatrix} 23 & 1 & -3 \\ 2 & 24 & 3 \\ -4 & 2 & 19 \end{bmatrix}$	M1 A1		≥ 5 correct for the M All 9 correct for the A
	$\mathbf{X}^2 - \mathbf{X} = 20\mathbf{I}$ i.e. $k = 20$	A1	3	Shown legitimately
	(ii) Mult^g. $\mathbf{X}^2 - \mathbf{X} = 20\mathbf{I}$ by $\mathbf{X}^{-1}$	M1		
	Re-arranging $\mathbf{X} - \mathbf{I} = 20\mathbf{X}^{-1}$ to get $\mathbf{X}^{-1} = \frac{1}{20}(\mathbf{X} - \mathbf{I})$	A1	2	Legitimately
(b)	$\mathbf{X}^{-1} = \frac{1}{20}\begin{bmatrix} 2 & 1 & -3 \\ 2 & 3 & 3 \\ -4 & 2 & -2 \end{bmatrix}$	B1		Noted or used
	$(\mathbf{XY})^{-1} = \mathbf{Y}^{-1}\mathbf{X}^{-1}$	M1		Incl. attempt at the multn.
	$\begin{bmatrix} 6 & 3 & -9 \\ 2 & -1 & 1 \\ 2 & 3 & 3 \end{bmatrix}$	A1	3	
	Total		**8**	

MFP4(cont)

Q	Solution	Marks	Total	Comments
5(a)	$\Pi_1 : 6x + 2y + 9z = 5$ $\Pi_2 : 10x - y - 11z = 4$	B1	1	Both
(b)	**Method 1**			
	E.g. $\Pi_1 + 2\,\Pi_2 \Rightarrow 13(2x - z = 1)$	M1		
	$\dfrac{x}{1} = \dfrac{z+1}{2} = \lambda$	M1 A1		Parametrisation attempt
	Using 1st two to find 3rd in terms of λ: $x = \lambda,\ z = 2\lambda - 1 \Rightarrow y = 7 - 12\lambda$	M1		
	$\mathbf{r} = \begin{bmatrix} 0 \\ 7 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix}$	A1	5	Any correct vector eqn.form
	Method 2			
	$(0, 7, -1)$ or $\left(\frac{7}{12},\ 0,\ \frac{1}{6}\right)$ or $\left(\frac{1}{2},\ 1,\ 0\right)$	(M1A1)		Finding a pt. on the line
	$\begin{bmatrix} 6 \\ 2 \\ 9 \end{bmatrix} \times \begin{bmatrix} 10 \\ -1 \\ -11 \end{bmatrix} = (\pm)13 \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix}$	(M1) (A1)		Finding a d.v.
	$\mathbf{r} = \begin{bmatrix} 0 \\ 7 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix}$	(A1)		Any correct vector eqn.form
(c)	Subst^g. $x = \lambda,\ y = 7 - 12\lambda,\ z = 2\lambda - 1$ into $5x + 3y + 11z = 28$	M1		
	Solving a linear eqn. in λ:	M1		
	$\lambda = -2$	A1		CAO
	$(-2, 31, -5)$	B1√	4	FT their λ in their line eqn.
(d)	$\mathbf{n} = \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix}$	B1√		FT when used as part of a plane eqn., *not* a line
	$d = \begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix} = 10$	M1		Attempted
	$\mathbf{r} \bullet \begin{bmatrix} 1 \\ -12 \\ 2 \end{bmatrix} = 10$	A1√	3	Any correct vector eqn. form FT
	Alt. $\mathbf{r} = \begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix} + \lambda \begin{bmatrix} 6 \\ 2 \\ 9 \end{bmatrix} + \begin{bmatrix} 10 \\ -1 \\ -11 \end{bmatrix}$	(M1) (A1) (A1)		Plane eqn. attempt Point + at least 1 d.v. All correct
	Total		**13**	

MFP4(cont)

Q	Solution	Marks	Total	Comments
6(a)	$\begin{bmatrix}1\\1\\-1\end{bmatrix}\bullet\begin{bmatrix}12\\15\\16\end{bmatrix}=12+15-16=11$ shown	B1	1	
(b) (i)	eqn., or use, of line incorporating p.v. $\begin{bmatrix}1\\1\\-1\end{bmatrix}$ and d.v. $\begin{bmatrix}12\\15\\16\end{bmatrix}$	M1		
	$\frac{x-1}{12}=\frac{y-1}{15}=\frac{z+1}{16}$	A1	2	
(ii)	$\sqrt{12^2+15^2+16^2}=25$	B1√		FT
	$\frac{12}{25},\ \frac{15}{25},\ \frac{16}{25}$	B1√	2	FT
(c)	Use of $\mathbf{r}=\begin{bmatrix}1\\1\\-1\end{bmatrix}+\lambda\begin{bmatrix}12\\15\\16\end{bmatrix}$ with $\lambda=\pm 2$	M1		
	(25, 31, 31) and (– 23, – 29, – 33)	A1 A1	3	
(d)	<u>**Method 1**</u>			
	$PQ = 5$	B1		
	Area $\Delta PMN = \frac{1}{2}\times PQ\times MN = 250$	M1 A1	3	(Since Q = midpt.MN)
	<u>**Method 2**</u>			
	$\overrightarrow{PM}=\begin{bmatrix}20\\30\\35\end{bmatrix}$, $\overrightarrow{PN}=\begin{bmatrix}-28\\-30\\-29\end{bmatrix}$	(M1)		Attempted
	$\overrightarrow{PM}\times\overrightarrow{PN}=(\pm)20\begin{bmatrix}9\\-20\\12\end{bmatrix}$ attempted within an area formula	(M1)		||gm. or Δ
	Area $\Delta PMN = 250$	(A1)		CAO
	Total		**11**	

MFP4(cont)

Q	Solution	Marks	Total	Comments
7(a)	Attempt at Char.Eqn.	M1		
	$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$	A3,2,1		One each following coefft.
	Attempt at (at least) one linear factor	M1		
	$(\lambda - 1)(\lambda - 4)^2 = 0$	A1		
	$\lambda = 1, 4, 4$	A1	7	
(b)	$\lambda = 1 \Rightarrow \begin{matrix} 2x - y + z = 0 \\ -x + 2y + z = 0 \\ x + y + 2z = 0 \end{matrix} \Rightarrow \alpha\begin{bmatrix}1\\1\\-1\end{bmatrix}$	M1 A1		Subst[g]. back and solving
	$\lambda = 4 \Rightarrow x + y - z = 0$	B1		
	Choosing any two independent vectors satisfying this	M1		
	E.g.s: $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}$	A1	5	
(c)	$\alpha\begin{bmatrix}1\\1\\-1\end{bmatrix}$ ≡ a line of invariant points	M1 A1		Invariant line LoIPs
	e.g. $\alpha\begin{bmatrix}1\\0\\1\end{bmatrix} + \beta\begin{bmatrix}0\\1\\1\end{bmatrix}$ ≡ an invariant plane	B1	3	
	Total		**15**	

MFP4(cont)

Q	Solution	Marks	Total	Comments
8(a)	Det(**M**) = – 1	B1		
	Magnitude = 1 $\Rightarrow$ area invariant	B1√		FT area s.f.
	– ve sign $\Rightarrow$ cyclic order of vertices is reversed OR "reflection" involved	B1	3	
(b)	**Method 1**			
	Char. Eqn.: $\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$	M1 A1		Finding and solving attempt
	Subst^g. back: $\lambda = 1 \Rightarrow y = \frac{1}{2}x$	M1 A1		
	and $\lambda = -1 \Rightarrow y = \frac{1}{4}x$	A1	5	
	Method 2			
	$\begin{bmatrix} -3 & 8 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ mx \end{bmatrix} = \begin{bmatrix} (8m-3)x \\ (3m-1)x \end{bmatrix}$	(M1)		Attempted
	Use of $y' = mx'$: $3m - 1 = 8m^2 - 3m$	(M1)		
	Solving a quadratic eqn. in $m = \frac{1}{4}, \frac{1}{2}$	(M1A1)		From $(4m - 1)(2m - 1) = 0$
	$p = \frac{1}{2}$ and $q = \frac{1}{4}$	(A1)		
(c)	**(i)** $p = \frac{1}{2} = \tan\theta$			
	$\Rightarrow \cos 2\theta = \frac{3}{5}$ and $\sin 2\theta = \frac{4}{5}$	M1		For these attempted and used in a reflection matrix
	$\mathbf{R} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$	A1	2	
	(ii) Use $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}\mathbf{S} = \begin{bmatrix} -3 & 8 \\ -1 & 3 \end{bmatrix}$	M1		FT their **R**
	S found using inverse matrix	M1		Or equivalent method
	$= \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}\begin{bmatrix} -3 & 8 \\ -1 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -13 & 36 \\ -9 & 23 \end{bmatrix}$	A1		
	Shear, parallel to $y = \frac{1}{2}x$	B1		CAO
	mapping (e.g.) $(1, 1) \rightarrow (4.6, 2.8)$	B1√	5	FT any pt. and its image
	Total		**15**	
	TOTAL		**75**	

AQA – Further pure 4 – Jan 2011 – Answers

Question 1:	Exam report
$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ x & y & z \\ x+y+z & x+y+z & x+y+z \end{vmatrix}$ $Combination: R_3^{'} = R_2 + R_3$ $\Delta = (x+y+z)\begin{vmatrix} 1 & 2 & 3 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = (x+y+z)\begin{vmatrix} 1 & 1 & 2 \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$ $Combinations: C_2^{'} = C_2 - C_1 \ \ and \ \ C_3^{'} = C_3 - C_1$ $\Delta = (x+y+z)\left((z-x) - 2(y-x)\right) = (x+y+z)(x-2y+z)$	This was a straightforward starter to the paper, and was generally found to be so by candidates. However, almost half of all candidates failed to score all 4 marks, due largely to arithmetical slips and carelessness with signs. This was mostly due to unhelpfully lengthy work on row- and/or column-operations after the (*x* + *y* + *z*) factor had been extracted, when the simplest approach was to just go ahead and expand the remaining determinant. This highlights the view that many of these candidates had had insufficient practice with handling determinants algebraically in order to arrive at the point when they might have a "feel" for what is a good approach to take at different stages. Moreover, even in this example, it was about 50-50 whether candidates added *R*3 into *R*2 or vice versa; the latter approach clearly producing a simpler determinant to continue to work with afterwards, if some thought had been given to it.

Question 2:	Exam report
$c = \lvert \mathbf{a} \times \mathbf{b} \rvert = abSin\theta \ \ and \ \ d = \lvert \mathbf{a}.\mathbf{b} \rvert = abCos\theta$ $c^2 + d^2 = a^2b^2Sin^2\theta + a^2b^2Cos^2\theta$ $c^2 + d^2 = a^2b^2(Cos^2\theta + Sin^2\theta) = a^2b^2$	This turned out to be much more troublesome for candidates than had been anticipated, with almost two-thirds of the candidature failing to score a single mark on the question, despite the injunction to consider the definitions of the two products. Even amongst those who did proceed as planned, a large number of them insisted on using modulus signs and/or a unit vector throughout, indicating a lack of grasp as to what should, and what should not, have been involved. Some of these misuses of both notation and understanding of vector and scalar matters were condoned.

Question 3:	Exam report
The system does not have a unique solution when the determinant of the associated matrix is 0 $\det = \begin{vmatrix} t & 2 & 3 \\ 2 & 3 & -t \\ 3 & 5 & t+1 \end{vmatrix} = t\begin{vmatrix} 3 & -t \\ 5 & t+1 \end{vmatrix} - 2\begin{vmatrix} 2 & -t \\ 3 & t+1 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}$ $\det = t(3t+3+5t) - 2(2t+2+3t) + 3(10-9)$ $\det = 8t^2 + 3t - 10t - 4 + 3 = 8t^2 - 7t - 1$ $\det = 0 \Leftrightarrow 8t^2 - 7t - 1 = 0$ $(8t+1)(t-1) = 0$ $t = 1 \ or \ t = -\frac{1}{8}$ $b) \ for \ t = 1, the \ system \ becomes$ $\begin{cases} x+2y+3z=a & l_1 \\ 2x+3y-z=b & l_2 \\ 3x+5y+2z=c & l_3 \end{cases} \Leftrightarrow \begin{cases} x+2y+3z=a & \\ y+7z=2a-b & (2l_1-l_2) \\ y+7z=3a-c & (3l_1-l_3) \end{cases}$ The system is consistent if and only if $2a - b = 3a - c$ $a + b - c = 0$	This was generally well done, with marks relatively high. It was surprising to find that those who failed to get any integer value for *t* generally didn't appear to go back and check their working. Weaker candidates often didn't seem to know how to go about establishing consistency.

Question 4:

$$a)\,i)\ \mathbf{X}^2-\mathbf{X}=\begin{pmatrix}23&1&-3\\2&24&3\\-4&2&19\end{pmatrix}-\begin{pmatrix}3&1&-3\\2&4&3\\-4&2&-1\end{pmatrix}=\begin{pmatrix}20&0&0\\0&20&0\\0&0&20\end{pmatrix}$$

$$\mathbf{X}^2-\mathbf{X}=20\mathbf{I}$$

$ii)$ By multiplying both sides by $\mathbf{X}^{-1}$, we have

$$\mathbf{X}^{-1}(\mathbf{X}^2-\mathbf{X})=20\mathbf{X}^{-1}\mathbf{I}\Leftrightarrow\quad \mathbf{X}-\mathbf{I}=20\mathbf{X}^{-1}$$

$$\mathbf{X}^{-1}=\frac{1}{20}(\mathbf{X}-\mathbf{I})=\frac{1}{20}\begin{pmatrix}2&1&-3\\2&3&3\\-4&2&-2\end{pmatrix}$$

$$b)\ (\mathbf{XY})^{-1}=\mathbf{Y}^{-1}\mathbf{X}^{-1}=\mathbf{Y}^{-1}\times\frac{1}{20}(\mathbf{X}-\mathbf{I})=$$

$$=\frac{1}{20}\begin{pmatrix}60&0&0\\0&0&-10\\0&20&0\end{pmatrix}\begin{pmatrix}2&1&-3\\2&3&3\\-4&2&-2\end{pmatrix}=\begin{pmatrix}6&3&-9\\2&-1&1\\2&3&3\end{pmatrix}$$

Exam report

This was another popular question, in that responses were generally at least partially successful. Even so, around half of all candidates failed to realise that the "hence" in part (a)(ii) meant that they were supposed to work algebraically with the result of part (a)(i) rather than go through the lengthy approach of finding $\mathbf{X}^{-1}$ directly. About the same proportion incorrectly opted for $(\mathbf{XY})^{-1}=\mathbf{X}^{-1}\mathbf{Y}^{-1}$. In addition to all these errors, a lot of marks were lost due to carelessness; mistakes made in calculating $\mathbf{X}^2$ were often not corrected, even when the candidates clearly failed to arrive at a multiple of **I** in part (a)(i).

Then, despite being told that $\mathbf{X}^{-1}$ could be found by $\frac{1}{20}(\mathbf{X}-\mathbf{I})$ in part (b), many made a slip somewhere in the working, and the correct $(\mathbf{XY})^{-1}$ appeared far less often than should have been the case, given the information given in the question.

Question 5:

$$a)\ \Pi_1:\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}6\\2\\9\end{pmatrix}=5\Leftrightarrow 6x+2y+9z=5$$

$$\Pi_2:\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}10\\-1\\-11\end{pmatrix}=4\Leftrightarrow 10x-y-11z=4$$

$b)$ A direction vector of the line of intersection is $\mathbf{u}=\begin{pmatrix}6\\2\\9\end{pmatrix}\times\begin{pmatrix}10\\-1\\-11\end{pmatrix}$

$\mathbf{u}=\begin{pmatrix}-13\\156\\-26\end{pmatrix}$, the vector $\begin{pmatrix}1\\-12\\2\end{pmatrix}$ is also a direction vector.

A point belonging to both plane satisfies the equations $\begin{cases}6x+2y+9z=5\\10x-y-11z=4\end{cases}$

$$\Leftrightarrow\begin{cases}26x-13z=13\quad(2l_2+l_1)\\6x+2y+9z=5\end{cases}\Leftrightarrow\begin{cases}2x-z=1\\6x+2y+9z=5\end{cases}$$

For example, the point P(0, 7, −1) belongs to both planes.

An equation of the line of intersection is $\mathbf{r}=\begin{pmatrix}0\\7\\-1\end{pmatrix}+t\begin{pmatrix}1\\-12\\2\end{pmatrix}$

$c)$ The point of intersection of the three planes is the point of intersection between the line and Π_3

$Q(t\ ,\ 7-12t\ ,\ -1+2t)$ *belongs to the plane with equation* $5x+3y+11z=28$

so $5t+3(7-12t)+11(-1+2t)=28$

$-9t+10=28\qquad t=-2$

and the coordinates of Q are $(-2, 31, -5)$

$d)$ Let's call Π_4 the plane perpendicular to Π_1 and Π_2 going through (4,1,9).

A normal vector to Π_4 is $\mathbf{i}-12\mathbf{j}+2\mathbf{k}$

(the direction vector of the line of intersection)

An equation of the plane Π_4 *is*: $\left(\mathbf{r}-\begin{pmatrix}4\\1\\9\end{pmatrix}\right).\begin{pmatrix}1\\-12\\2\end{pmatrix}=0\Leftrightarrow\mathbf{r}.\begin{pmatrix}1\\-12\\2\end{pmatrix}=10$

Exam report

Around a third of the candidature attempted little beyond part (a), which was very surprising.

Of the majority who did proceed well into the question, part(b) was usually done well, although there were many sign errors that arose in the vector product of the two normals. Another surprise was the lack of appreciation that there was a factor of 13 which could be ignored here.

Of those who realised in part(c) that they simply had to substitute part(b)'s answer into Π_3's equation, a slight majority had incorrect components to work with, and this made follow through marking difficult.

In part(d), only a very few spotted that they could answer this without any need to rely on previous answers, as a plane equation in the form $\mathbf{r}=\mathbf{a}+\lambda\,\mathbf{d}_1+\mu\,\mathbf{d}_2$ could be written straight down using the two normals given in the question. The most popular form was, of course, $\mathbf{r}\,.\,\mathbf{n}=d$, although a lot of candidates opted for a line equation here instead.

Question 6:

$a)$ Let's work out $\begin{pmatrix}1\\1\\-1\end{pmatrix}.\begin{pmatrix}12\\15\\16\end{pmatrix}=12+15-16=11$

Q belongs to the plane Π.

$b) i)$ A direction vector of the line is $\mathbf{n}=12\mathbf{i}+15\mathbf{j}+16\mathbf{k}$

and cartesian equations of the line are: $\frac{x-1}{12}=\frac{y-1}{15}=\frac{z+1}{16}$

$ii) |\mathbf{n}|=\sqrt{12^2+15^2+16^2}=25$

The direction cosines are : $\frac{12}{25}, \frac{15}{25}=\frac{3}{5}, \frac{16}{25}$

$c)$ A vector equation of l is : $\mathbf{r}=\begin{pmatrix}1\\1\\-1\end{pmatrix}+t\begin{pmatrix}12\\15\\16\end{pmatrix}$

A point P on the line has coordinates $P(1+12t, 1+15t, -1+16t)$

The distance PQ is 50 i.e $|\overrightarrow{PQ}|=50$

$$\overrightarrow{PQ}=\begin{pmatrix}12t\\15t\\16t\end{pmatrix} \quad and \quad |\overrightarrow{PQ}|=\sqrt{(12t)^2+(15t)^2+(16t)^2}=50$$

$$625t^2=50^2 \qquad t^2=4$$

$t=2$ or $t=-2$

This gives $M(25,31,31)$ and $N(-23,-29,-33)$

$d)$ In the triangle PMN, the base MN =100 and the height is PQ

$\overrightarrow{PQ}=-4\mathbf{i}+3\mathbf{k}$ and $PQ=\sqrt{16+9}=5$

The area of the triangle is $\frac{1}{2}NM\times PQ=\frac{1}{2}\times100\times5=250$

Exam report

Around a quarter of all candidates made attempts only at part (a), for just the one mark, and around half the candidates failed to proceed beyond part (b). Even amongst these, there was a large number of candidates who apparently didn't know the difference between vector and cartesian equations, with many others not happy to work with the given normal vector. Of that half of the candidature that did proceed beyond part (b), most of them decided that "a distance of 50 units" meant that λ was $\pm$ 50, rather than $\pm$ 2. They had failed to realise that they had just worked out that the direction vector 12**i** + 15**j** + 16**k** was 25 units long. This was very disappointing. Even in the final part of the question, despite being given the coordinates of *P* and *Q* and the information that the line *PQ* was the height of a right-angled triangle with base 100, only about five candidates (of 336) appreciated that they could work out the area of the triangle just as it was. Even more disappointing than this was that so few could employ alternative vector methods to get the correct answer, mostly due to not having had much success with parts (a) to (c).

Question 7:

$a)$ Let's work out $\det(\mathbf{Y}\text{-}\lambda\mathbf{I})$

$$\det(\mathbf{Y}\text{-}\lambda\mathbf{I})=\begin{vmatrix}3-\lambda & -1 & 1\\ -1 & 3-\lambda & 1\\ 1 & 1 & 3-\lambda\end{vmatrix}=\begin{vmatrix}4-\lambda & -1 & 0\\ -4+\lambda & 3-\lambda & 4-\lambda\\ 0 & 1 & 4-\lambda\end{vmatrix}$$

We did : $C_1'=C_1-C_2$ *and* $C_3'=C_2+C_3$

$$\det(\mathbf{Y}\text{-}\lambda\mathbf{I})=(4-\lambda)^2\begin{vmatrix}1 & -1 & 0\\ -1 & 3-\lambda & 1\\ 0 & 1 & 1\end{vmatrix}=(4-\lambda)^2\begin{vmatrix}1 & -1 & 0\\ 0 & 2-\lambda & 1\\ 0 & 1 & 1\end{vmatrix}$$

$\det(\mathbf{Y}\text{-}\lambda\mathbf{I})=(4-\lambda)^2(2-\lambda-1)=(4-\lambda)^2(1-\lambda)$

$\det(\mathbf{Y}\text{-}\lambda\mathbf{I})=0 \Leftrightarrow \lambda=4$ (*repeated value*) *or* $\lambda=1$

$b)$ To find the eigenvectors we need to solve $(\mathbf{Y}\text{-}\lambda\mathbf{I})\begin{pmatrix}x\\y\\z\end{pmatrix}=0$

$$For\ \lambda=1, \begin{pmatrix}2 & -1 & 1\\ -1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=0 \Leftrightarrow \begin{cases}2x-y+z=0\\ -x+2y+z=0\\ x+y+2z=0\end{cases} \Leftrightarrow \begin{cases}3x-3y=0\\ -x+2y+z=0\\ x+y+2z=0\end{cases}$$

$\Leftrightarrow \begin{cases}x=y\\ z=-x\end{cases}$ An eigenvector is $\mathbf{u}_1=\mathbf{i}+\mathbf{j}-\mathbf{k}$

$$For\ \lambda=4, \begin{pmatrix}-1 & -1 & 1\\ -1 & -1 & 1\\ 1 & 1 & -1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=0 \Leftrightarrow \begin{cases}-x-y+z=0\\ -x-y+z=0\\ x+y-z=0\end{cases} \Leftrightarrow \{x+y-z=0$$

Eigenvectors are $\mathbf{u}_2=\mathbf{i}+\mathbf{j}-2\mathbf{k}$ *and* $\mathbf{u}_3=\mathbf{j}+\mathbf{k}$ (two independent vectors)

c) The line going through O with direction vector $\mathbf{u}_1$ is a line of invariant points.

The plane going through O with direction vectors $\mathbf{u}_2$ *and* $\mathbf{u}_3$ is an invariant plane.

Exam report

This question produced many more at least partially successful attempts. Nonetheless, sign errors and arithmetical slips abounded and there was, yet again, very little evidence of tracking back to check obviously incorrect working and answers. It was only a very small minority who appreciated that a repeated eigenvalue should lead to an invariant plane, which needs two representative eigenvectors.
Uncertainty in parts (b) and (c) often led to marks not exceeding 10 out of the 15 available, even amongst the attempts of the better candidates.

Question 8:

$a)\ \det(\mathbf{M}) = \begin{vmatrix} -3 & 8 \\ -1 & 3 \end{vmatrix} = -9+8 = -1$

The area is invariant through the tranformation but the cyclic order of the vertices is reversed.

b) • Any point $P(x, px)$ is invariant, this mean that $\mathbf{M}\begin{pmatrix} x \\ px \end{pmatrix} = \begin{pmatrix} x \\ px \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} -3 & 8 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ px \end{pmatrix} = \begin{pmatrix} (-3+8p)x \\ (-1+3p)x \end{pmatrix} = \begin{pmatrix} x \\ px \end{pmatrix}$

We need to have $-3+8p=1$ AND $-1+3p=p$

$p=\frac{1}{2}\ \ AND\ \ p=\frac{1}{2}$ (the system is consistent)

The line of invariant points has equation $y=\frac{1}{2}x$

• The line $y=qx$ is an invariant line. This means that for all x:

$\mathbf{M}\begin{pmatrix} x \\ qx \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ *with* $y'=qx'$.

$\mathbf{M}\begin{pmatrix} x \\ qx \end{pmatrix} = \begin{pmatrix} (-3+8q)x \\ (-1+3q)x \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

$y'=qx'$ *gives* $(-1+3q)x = q(-3+8q)x$

$3q-1=-3q+8q^2 \Leftrightarrow 8q^2-6q+1=0$

$\Leftrightarrow (4q-1)(2q+1)=0 \Leftrightarrow q=\frac{1}{4}$ *or* $q=\frac{1}{2}=p$

$c)\ i)\ y=\frac{1}{2}x = Tan\theta\, x$

$Cos2\theta = 2Cos^2\theta - 1 = \frac{2}{1+Tan^2\theta} - 1 = \frac{3}{5}$

$Sin2\theta = \sqrt{1-Cos^2\theta} = \frac{4}{5}$

The matrix $\mathbf{R}$, reflection in the line $y=\frac{1}{2}x$, *is*

$\mathbf{R} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$

$ii)\ \mathbf{M} = \mathbf{RS}$ *and because* $\mathbf{R}^{-1} = \mathbf{R}$ (*reflection*)

then $\mathbf{RM} = \mathbf{S} \Leftrightarrow \mathbf{S} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} -3 & 8 \\ -1 & 3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -13 & 36 \\ -9 & 23 \end{pmatrix}$

$\mathbf{S}$ represents a shear parallel to the line $y=\frac{1}{2}x$

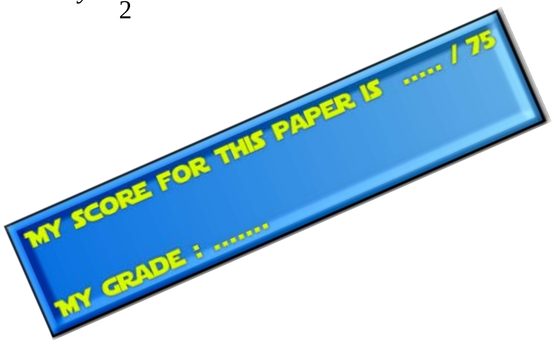
which maps $(1,0)$ onto $(-\frac{13}{5}, -\frac{9}{5})$

Exam report

In part(a), many candidates realised the significance of both the sign and magnitude of the determinant of the transformation matrix, although there are still far too many who try to describe area scale factors without using the key word "area".

In part(b), almost all attempts managed to find or verify the given value of *p*, but most who did go on to try and find a value for *q* as well simply did what amounted to the same working again. The more successful bids found the eigenvalues first and then deduced *p* and *q*.

Attempts at part(c)(i) fell almost equally into the three categories of correct, incorrect – usually a 45° or 90° rotation matrix – or nothing at all. In part(c)(ii), however, candidates' lack of exam-readiness was once again markedly to the fore. The order of the two matrices was the right way round only about 50% of the time. More significantly the great majority decided that the best way to find a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -3 & 8 \\ -1 & 3 \end{pmatrix}$ (either way round on the LHS) was to multiply out and solve four simultaneous equations in the four unknowns *a*, *b*, *c*, *d* rather than pre-mutliply by an inverse matrix. Needless to say, most such attempts went wrong somewhere with the accompanying arithmetic and/or algebra, often due to pre-multiplying on one side of the matrix equation whilst post-multiplying on the other. Even more disappointing was the almost total lack of success of those attempts which did try to employ the inverse of a reflection matrix – candidates should have realised that such a matrix is self-inverse. In describing the shear in detail few candidates were very clear as to what was wanted. Even amongst those who reached the end of the question still with the opportunity to offer an answer, most didn't. Rather bizarrely, many offerings from candidates who did state something opted for a shear either parallel to one of the coordinate axes or parallel to (their) *y = qx*, despite the rather obvious flagging of $y=\frac{1}{2}x$ at several stages of the question.



Grade boundaries

Grade		A*	A	B	C	D	E
Mark	Max 75	63	55	47	40	33	26