General Certificate of Education
Advanced Level Examination
January 2010

Mathematics MFP4

Unit Further Pure 4

Monday 25 January 2010 9.00 am to 10.30 am

For this paper you must have:
- a 12-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The **Examining Body** for this paper is AQA. The **Paper Reference** is MFP4.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 The 2×2 matrix $\mathbf{M}$ represents the plane transformation T. Write down the value of $\det \mathbf{M}$ in each of the following cases:

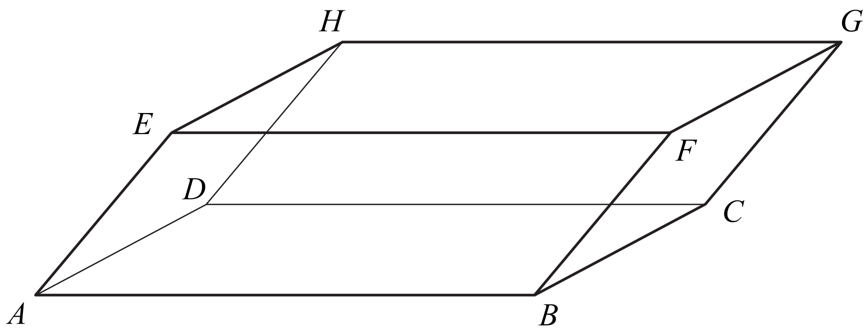
(a) T is a rotation;

(b) T is a reflection;

(c) T is a shear;

(d) T is an enlargement with scale factor 3. *(4 marks)*

2 The diagram shows the parallelepiped *ABCDEFGH*.



The position vectors of *A*, *B*, *C*, *D* and *E* are, respectively,

$$\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -3 \\ 10 \\ 4 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} -7 \\ 10 \\ 7 \end{bmatrix} \quad \text{and} \quad \mathbf{e} = \begin{bmatrix} 3 \\ 4 \\ 10 \end{bmatrix}$$

(a) Show that the area of *ABCD* is 37. *(4 marks)*

(b) Find the volume of *ABCDEFGH*. *(2 marks)*

(c) Deduce the distance between the planes *ABCD* and *EFGH*. *(2 marks)*

3 The matrices **A** and **B** are defined in terms of a real parameter t by

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & t & 4 \\ 3 & 2 & -1 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 15 & -4 & -1 \\ -2t & 4 & 2 \\ 17 & -4 & -3 \end{bmatrix}$$

(a) Find, in terms of t, the matrix **AB** and deduce that there exists a value of t such that **AB** is a scalar multiple of the 3×3 identity matrix **I**. *(5 marks)*

(b) For this value of t, deduce $\mathbf{A}^{-1}$. *(2 marks)*

4 (a) Determine the two values of k for which the system of equations

$$\begin{aligned} x - 2y + kz &= 5 \\ (k+1)x + 3y &= k \\ 2x + y + (k-1)z &= 3 \end{aligned}$$

does not have a unique solution. *(4 marks)*

(b) Show that this system of equations is consistent for one of these values of k, but is inconsistent for the other.

(You are not required to find any solutions to this system of equations.) *(8 marks)*

5 The plane transformations $\mathrm{T_A}$ and $\mathrm{T_B}$ are represented by the matrices **A** and **B** respectively, where $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 5 \\ 5 & 13 \end{bmatrix}$.

(a) Find the equation of the line which is the image of $y = 2x + 1$ under $\mathrm{T_A}$. *(3 marks)*

(b) The rectangle $PQRS$, with area 4.5 cm^2, is mapped onto the parallelogram $P'Q'R'S'$ under $\mathrm{T_B}$. Determine the area of $P'Q'R'S'$. *(2 marks)*

(c) The transformation $\mathrm{T_C}$ is the composition

'$\mathrm{T_B}$ followed by $\mathrm{T_A}$'

By finding the matrix which represents $\mathrm{T_C}$, give a full geometrical description of $\mathrm{T_C}$. *(5 marks)*

Turn over for the next question

6 (a) Find the value of p for which the planes with equations

$$\mathbf{r} \cdot \begin{bmatrix} 6 \\ -3 \\ 2 \end{bmatrix} = 42 \quad \text{and} \quad \mathbf{r} \cdot \begin{bmatrix} 4p+1 \\ p-2 \\ 1 \end{bmatrix} = -7$$

(i) are perpendicular; *(3 marks)*

(ii) are parallel. *(3 marks)*

(b) In the case when $p = 4$:

(i) write down a cartesian equation for each plane; *(2 marks)*

(ii) find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, an equation for l, the line of intersection of the planes. *(6 marks)*

(c) Determine a vector equation, in the form $\mathbf{r} = \mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$, for the plane which contains l and which passes through the point (30, 7, 30). *(2 marks)*

7 (a) It is given that $\Delta = \begin{vmatrix} 16-q & 5 & 7 \\ -12 & -1-q & -7 \\ 6 & 6 & 10-q \end{vmatrix}$.

(i) By using row operations on the first two rows of Δ, show that $(4-q)$ is a factor of Δ. *(2 marks)*

(ii) Express Δ as the product of three linear factors. *(4 marks)*

(b) It is given that $\mathbf{M} = \begin{bmatrix} 16 & 5 & 7 \\ -12 & -1 & -7 \\ 6 & 6 & 10 \end{bmatrix}$.

(i) Verify that $\begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix}$ is an eigenvector of $\mathbf{M}$ and state its corresponding eigenvalue. *(3 marks)*

(ii) For each of the other two eigenvalues of $\mathbf{M}$, find a corresponding eigenvector. *(7 marks)*

(c) The transformation T has matrix $\mathbf{M}$. Write down cartesian equations for any one of the invariant lines of T. *(2 marks)*

END OF QUESTIONS

Key to mark scheme and abbreviations used in marking

M	mark is for method
m or dM	mark is dependent on one or more M marks and is for method
A	mark is dependent on M or m marks and is for accuracy
B	mark is independent of M or m marks and is for method and accuracy
E	mark is for explanation

√or ft or F	follow through from previous incorrect result	MC	mis-copy
CAO	correct answer only	MR	mis-read
CSO	correct solution only	RA	required accuracy
AWFW	anything which falls within	FW	further work
AWRT	anything which rounds to	ISW	ignore subsequent work
ACF	any correct form	FIW	from incorrect work
AG	answer given	BOD	given benefit of doubt
SC	special case	WR	work replaced by candidate
OE	or equivalent	FB	formulae book
A2,1	2 or 1 (or 0) accuracy marks	NOS	not on scheme
$-x$ EE	deduct x marks for each error	G	graph
NMS	no method shown	c	candidate
PI	possibly implied	sf	significant figure(s)
SCA	substantially correct approach	dp	decimal place(s)

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded. However, there are situations in some units where part marks would be appropriate, particularly when similar techniques are involved. Your Principal Examiner will alert you to these and details will be provided on the mark scheme.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

Q	Solution	Marks	Total	Comments
1(a)	1	B1		
(b)	– 1	B1		
(c)	1	B1		
(d)	9	B1	4	
	Total		**4**	
2(a)	$\overrightarrow{AB}\times\overrightarrow{AD}=\begin{bmatrix}4\\0\\-3\end{bmatrix}\times\begin{bmatrix}-8\\7\\3\end{bmatrix}$	M1		Attempt at vector product of any two suitable vectors
	$=\begin{bmatrix}21\\12\\28\end{bmatrix}$	A1		
	$\sqrt{21^2+12^2+28^2}$	M1		Magnitude of this attempted
	$= 37$	A1	4	ft on minus signs only AG
(b)	$\overrightarrow{AB}\times\overrightarrow{AD}\bullet\overrightarrow{AE}=\begin{bmatrix}21\\12\\28\end{bmatrix}\bullet\begin{bmatrix}2\\1\\6\end{bmatrix}=222$	M1		Attempt at scalar triple product (or ≡) on any 3 suitable vectors
		A1	2	ft
(c)	Distance = (b) / 37 = 6	M1A1	2	ft; must be "deduced"
	Total		**8**	
3(a)	$\begin{bmatrix}32-4t & 0 & 0\\98-2t^2 & 4t-24 & 2t-14\\28-4t & 0 & 4\end{bmatrix}$	M1 A1 A1		Decent attempt at **AB** ≥ 5 correct all correct
	$t = 7$	A1		Allow this ft if only 1 or 2 elements of **AB** incorrect
	AB = 4**I**	A1	5	Must be from **AB** completely correct
(b)	$\mathbf{A}^{-1}=\frac{1}{4}\begin{bmatrix}15 & -4 & -1\\-14 & 4 & 2\\17 & -4 & -3\end{bmatrix}$	B1 B1	2	ft 1/det if related to **B** CAO; must be "deduced" NB "$\frac{1}{4}\mathbf{B}$" scores B1 only
	Total		**7**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
4(a)	$\begin{vmatrix} 1 & -2 & k \\ k+1 & 3 & 0 \\ 2 & 1 & k-1 \end{vmatrix} = 3k^2 - 2k - 5$	M1 A1		Attempt at det. of coefft. mtx. Correct
		M1		Setting det. = 0 and solving for k
	$k = \frac{5}{3}, -1$	A1	4	
(b)	$k = \frac{5}{3} \Rightarrow \begin{array}{rcl} 5 & = & x - 2y + \frac{5}{3}z \\ \frac{5}{3} & = & \frac{8}{3}x + 3y \\ 3 & = & 2x + y + \frac{2}{3}z \end{array}$	B1		ft
		M1		Eliminating one variable twice
	$8x + 9y = 5$ / $15y - 8z = -21$ / $5x + 3z = 11$	A1;A1		Correct eqn. once; twice
	$k = -1 \Rightarrow \begin{array}{rcl} 5 & = & x - 2y - z \\ -1 & = & 3y \\ 3 & = & 2x + y - 2z \end{array}$	B1		ft
		M1		Eliminating one variable twice
	$x - z = \frac{13}{3}$ / $2x - 2z = \frac{10}{3}$ / $5x - 5z = 11$			
	OR	A1		Inconsistency correctly shown
	$y = -\frac{1}{3}$ and $y = -\frac{7}{5}$ found	A1	8	
	Total		**12**	
5(a)	$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} x \\ 2x+1 \end{bmatrix} = \begin{bmatrix} x-1 \\ -x+2 \end{bmatrix}$	M1 A1		Use of $\begin{bmatrix} x \\ 2x+1 \end{bmatrix}$; either term correct
	giving $y = 1 - x$	A1	3	CSO
(b)	Finding det(**B**) (= 1) and mult^{g}. by 4.5 = 4.5 cm^2	M1 A1	2	ft
(c)	$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 5 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$	M1 A1		AB attempted
	This is a shear parallel to the x-axis	M1 A1		
	mapping (eg) (1, 1) to (3, 1)	A1	5	Any point not on Ox and its image
	Total		**10**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
6(a)(i)	$\begin{bmatrix}6\\-3\\2\end{bmatrix}\bullet\begin{bmatrix}4p+1\\p-2\\1\end{bmatrix}=0$	M1		Equating dot product to zero
	$24p+6-3p+6+2=0$	M1		Solving a linear eqn. in t
	$p=-\frac{2}{3}$	A1	3	CAO
(ii)	$\begin{bmatrix}6\\-3\\2\end{bmatrix}=m\begin{bmatrix}4p+1\\p-2\\1\end{bmatrix}$	M1		
	$m=2$, $p=\frac{1}{2}$	A1,A1	3	
	ALT $\begin{bmatrix}6\\-3\\2\end{bmatrix}\times\begin{bmatrix}4p+1\\p-2\\1\end{bmatrix}=\mathbf{0}$ $\begin{bmatrix}1-2p\\8p-4\\18p-9\end{bmatrix}$	(M1A1)		
	$p=\frac{1}{2}$	(A1)		
(b)(i)	$6x-3y+2z=42$, $17x+2y+z=-7$	B1,B1	2	
(ii)	eg 2.② – ①: $7[4x+y=-8]$	M1A1		Eliminating one variable; correct
	$\frac{x+2}{-1}=\frac{y}{4}=\lambda$	M1		Parametrisation attempt
	Substituting back to find 3rd variable $z=9\lambda+27$	m1 A1		
	$\mathbf{r}=\begin{bmatrix}0\\-8\\9\end{bmatrix},\begin{bmatrix}1\\-12\\0\end{bmatrix}$ **or** $\begin{bmatrix}-2\\0\\27\end{bmatrix}+\lambda\begin{bmatrix}-1\\4\\9\end{bmatrix}$	B1	6	ft (any pt. on l)
	OR method for finding the d.v. $\begin{bmatrix}-1\\4\\9\end{bmatrix}$	(M2A1)		
	Method for finding any pt. on l Putting them together as a line eqn.	(M1A1) (B1)		 ft
(c)	$\mathbf{r}=\mathbf{a}+u\,\mathbf{d}_1+v\,\mathbf{d}_2$ where $\mathbf{a}$ is an pt. on plane $\mathbf{d}_1$ = any d.v. $\mathbf{d}_2=\begin{bmatrix}30\\7\\30\end{bmatrix}$ – any $\mathbf{a}$ in plane		2	 ft their previous $\mathbf{a}$; or (30, 7, 30) ft their previous d.v. ft if is clear where it has come from
	Total		**16**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
7(a)(i)	$R_1' = R_1 + R_2$ or $R_2' = R_2 + R_1$	M1		
	leading to $R_{1/2} = (4-q \quad 4-q \quad 0)$	A1	2	
(ii)	$\Delta = (4-q)\begin{vmatrix} 1 & 1 & 0 \\ -12 & -1-q & -7 \\ 6 & 6 & 10-q \end{vmatrix}$			
	$= (4-q)\begin{vmatrix} 0 & 1 & 0 \\ q-11 & -1-q & -7 \\ 0 & 6 & 10-q \end{vmatrix}$	M1		By $C_1' = C_1 - C_2$ (eg)
	$= (4-q)(q-11)\begin{vmatrix} 0 & 1 & 0 \\ 1 & -1-q & -7 \\ 0 & 6 & 10-q \end{vmatrix}$	A1		2nd linear factor correct
	$= (4-q)(q-11) \times \ldots$	M1		Full attempt at 3rd factor
	$= (4-q)(q-11)(q-10)$	A1	4	
(b)(i)	$\begin{bmatrix} 16 & 5 & 7 \\ -12 & -1 & -7 \\ 6 & 6 & 10 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 8 \\ 20 \\ -28 \end{bmatrix}$	M1A1		
	$= 4\begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix}$ so that $\lambda = 4$	A1	3	
(ii)	$\lambda = 10, 11$ noted or used	B1		ft
	$\lambda = 10 \Rightarrow \begin{aligned} 0 &= 6x+5y+7z \\ 0 &= -12x-11y-7z \\ 0 &= 6x+6y \end{aligned}$	B1		
	Using $y = -x$ to get $z \sim x / y$	M1		
	Eigenvector $(-7, 7, 1)$	A1		Any non-zero multiple will do
	$\lambda = 11 \Rightarrow \begin{aligned} 0 &= 5x+5y+7z \\ 0 &= -12x-12y-7z \\ 0 &= 6x+6y-z \end{aligned}$	B1		
	Either $y = -x$ or $z = 0$	M1		
	Eigenvector $(1, -1, 0)$	A1	7	Any non-zero multiple will do
(c)	$\frac{x}{2} = \frac{y}{5} = \frac{z}{-7}$, $\quad \frac{x}{-7} = \frac{y}{7} = z$	M1		Any line eqns.
	or $x = -y$, $z = 0$	A1	2	any one ft correct
	Total		**18**	
	TOTAL		**75**	

AQA – Further pure 4 – Jan 2010 – Answers

Question 1:	Exam report
1) The moduls of the determinant of the matrix is the area scale factor of the transformation. $a)$ Rotation : $\det(\mathbf{M}) = 1$ $b)$ Reflection: $\det(\mathbf{M}) = -1$ $c)$ Shear: $\det(\mathbf{M}) = 1$ d) Enlargement: $\det(\mathbf{M}) = 3^2 = 9$	This was a straightforward starter to the paper, yet one that still required a bit of thought; especially with parts (b) and (d), where, respectively, the minus sign and the extra factor of 3 were often overlooked.

Question 2:	Exam report
$a)\ Area\ \ ABCD = \left\|\overrightarrow{AB} \times \overrightarrow{AD}\right\|$ $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$ *and* $\overrightarrow{AD} = \mathbf{d} - \mathbf{a} = \begin{pmatrix} -8 \\ 7 \\ 3 \end{pmatrix}$ $\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & -3 \\ -8 & 7 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -3 \\ 7 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4 & -3 \\ -8 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4 & 0 \\ -8 & 7 \end{vmatrix}\mathbf{k}$ $\overrightarrow{AB} \times \overrightarrow{AD} = 21\mathbf{i} + 12\mathbf{j} + 28\mathbf{k}$ $Area\ \ ABCD = \left\|\overrightarrow{AB} \times \overrightarrow{AD}\right\| = \sqrt{(21)^2 + (12)^2 + (28)^2} = 37$ $b)$ The volume of the parallelepiped is $\mathrm{V} = \left\|\overrightarrow{AE}.\left(\overrightarrow{AB} \times \overrightarrow{AD}\right)\right\|$ $\overrightarrow{AE} = \mathbf{e} - \mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}$ *and* $\overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix} 21 \\ 12 \\ 28 \end{pmatrix}$ $\overrightarrow{AE}.\left(\overrightarrow{AB} \times \overrightarrow{AD}\right) = \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}.\begin{pmatrix} 21 \\ 12 \\ 28 \end{pmatrix} = 42 + 12 + 168 = 222$ $Volume = 222$ $c)$ The distance between the two planes is the height of the parallelepiped, h. $Volume\ of\ \ parallelepiped = Area\ \ of\ \ the\ base \times height$ $= 37 \times h = 222$ $so\ \ h = 6$	Apart from a small minority of candidates who worked with the position vectors of *A*, *B* etc rather than the vectors $\overrightarrow{AB}, \overrightarrow{AD}$ etc, this was usually found quite straightforward — at least up until part (c), when the significance of the word “deduce” was almost universally ignored. Its original intention was to spare candidates from lengthy amounts of working, and point them in the direction of the obvious method, reinforced by the inclusion of a diagram. Remarkably few candidates spotted that the required distance was simply part (b)’s answer divided by part (a)’s, which was a shame for those who went ahead with other methods, as they received no credit. That said, almost no-one who tried one of these other methods obtained the right answer anyway.

Question 3:

$$\mathbf{A}=\begin{bmatrix}1&2&1\\2&t&4\\3&2&-1\end{bmatrix}\ \text{and}\ \mathbf{B}=\begin{bmatrix}15&-4&-1\\-2t&4&2\\17&-4&-3\end{bmatrix}$$

$$\mathbf{AB}=\begin{bmatrix}15-4t+17&-4+8-4&-1+4-3\\30-2t^2+68&-8+4t-16&-2+2t-12\\45-4t-17&-12+8+4&-3+4+3\end{bmatrix}=\begin{bmatrix}32-4t&0&0\\98-2t^2&4t-24&2t-14\\28-4t&0&4\end{bmatrix}$$

If $\mathbf{AB}=kI$ then k must be 4.

Is there a value of t for which

$32-4t=4$, $4t-24=4$, $2t-14=0$, $98-2t^2=0$ and $28-4t=0$?

When $t=7$ all the equations are satisfied.

b) If $\mathbf{AB}=4\mathbf{I}$ then $\mathbf{A}^{-1}\mathbf{AB}=4\mathbf{A}^{-1}\mathbf{I}$

$$\mathbf{B}=4\mathbf{A}^{-1} \qquad \mathbf{A}^{-1}=\frac{1}{4}\mathbf{B}=\frac{1}{4}\begin{bmatrix}15&-4&-1\\-14&4&2\\17&-4&-3\end{bmatrix}$$

Exam report

The matrix multiplication was generally handled very well, apart from a small but noticeable number of candidates who insisted on attempting **BA** despite the question. The only other obstacle to a completely correct part (a) was found amongst those candidates who had the odd one or two incorrect elements, which they had failed to notice and correct due to a lack of a check, meaning that $t = 7$ did not actually give 4**I** consistently for their **AB**. As with question 2, the word "deduce" in part (b) was almost totally ignored, and alternative methods for finding an inverse, often taking up lots of time, received no credit.

Question 4:

$a)$ The system of equations is equivalent to the matrix equation:

$$\begin{bmatrix}1&-2&k\\k+1&3&0\\2&1&k-1\end{bmatrix}\times\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}5\\k\\3\end{bmatrix}$$

The system does not have a unique solution means that the determinant of the matrix is 0

$$\begin{vmatrix}1&-2&k\\k+1&3&0\\2&1&k-1\end{vmatrix}=0 \text{ this gives } 3(k-1)+2(k+1)(k-1)+k(k+1-6)=0$$

$$3k-3+2k^2-2+k^2-5k=0$$

$$3k^2-2k-5=0$$

$$(3k-5)(k+1)=0$$

$$k=\frac{5}{3} \text{ or } k=-1$$

$b)$ • for $k=-1$, the system becomes

$$\begin{cases}x-2y-z=5 & l_1\\3y=-1 & l_2\\2x+y-2z=3 & l_3\end{cases}\quad \text{after combining}\quad\begin{cases}x-2y-z=5\\3y=-1\\3y=-7 & (-2l_1+l_3)\end{cases}$$

The system is inconsistent for $k=-1$

• for $k=\frac{5}{3}$, the system becomes

$$\begin{cases}x-2y+\frac{5}{3}z=5 & l_1\\\frac{8}{3}x+3y=\frac{5}{3} & l_2\\2x+y+\frac{2}{3}z=3 & l_3\end{cases}\quad \text{Multiplying by 3}\quad\begin{cases}3x-6y+5z=15 & L_1\\8x+9y=5 & L_2\\6x+3y+2z=9 & L_3\end{cases}$$

Eliminating z by combining the equations: $\begin{cases}3x-6y+5z=15\\8x+9y=5 & L_2'\\24x+27y=15 & L_3'=5L_3-2L_1(=3L_2')\end{cases}$

The system is consistent.

Exam report

This question was surprisingly well attempted, as the algebra that goes with the systems of equations work is usually found tough enough to guarantee lots of mistakes. Almost all candidates found the determinant of the coefficient matrix in part (a) and solved the resulting quadratic equation with ease. Other approaches generally got nowhere. The work in part (b) was competently attempted by the majority, apart from silly arithmetical slips which again probably would have been noticed had a very quick check been made.

Question 5:	Exam report
Any point on the line $y = 2x + 1$ *has vector position* $\begin{pmatrix} x \\ 2x+1 \end{pmatrix}$ The image of this point through T_A *is* $\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \times \begin{pmatrix} x \\ 2x+1 \end{pmatrix}$ $= \begin{pmatrix} 3x-2x-1 \\ -5x+4x+2 \end{pmatrix} = \begin{pmatrix} x-1 \\ -x+2 \end{pmatrix}$ $\begin{cases} x' = x-1 \\ y' = -x+2 \end{cases} \quad \begin{cases} x = 1+x' \\ y' = -1-x'+2 = -x'+1 \end{cases}$ The image of the line $y = 2x + 1$ *is the line* $y = -x + 1$ $b)$ The area scale factor of the transformation T_B *is* $\det(\mathbf{B})$ $\begin{vmatrix} 2 & 5 \\ 5 & 13 \end{vmatrix} = 26 - 25 = 1$ So the area of P'Q'R'S' is $4.5 \times 1 = 4.5\text{cm}^2$ $c)\ T_C = T_A \times T_B = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & 5 \\ 5 & 13 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ The transformation is a shear parallel to the x-axis which maps (1,1) into (3,1)	Rather bizarrely, the final bit of part (a) proved to be the most common difficulty on the paper. Having found the image of (x, $2x + 1$) to be ($x - 1$, $2 - x$), hardly anyone managed to deduce that the equation of the image-line was $y = 1 - x$. Parts (b) and (c) were handled very well indeed, although a few candidates tried **BA** rather than **AB** in part (c). The message as to what is an acceptable way to describe a shear has been stated often in recent years' reports, and this message has clearly been picked up by nearly all centres: the mapping of a point (not on the line of invariant points) to its image was noted far more regularly than in previous sessions, though there are still some rather spurious references to scale factor — 'spurious' in that it is not easy to assign to it an obvious significance or role in the proceedings; indeed, it is not easy even to decide what it is in cases where the shear is not parallel to one of the coordinate axes.

Question 6: | **Exam report**

a) i) The plane are perpendicular when the normal vectors are orthogonal: $\mathbf{n}.\mathbf{n}'=0$

$$\begin{pmatrix}6\\-3\\2\end{pmatrix}.\begin{pmatrix}4p+1\\p-2\\1\end{pmatrix}=0 \Leftrightarrow 24p+6-3p+6+2=0$$

$$\Leftrightarrow 21p+14=0 \quad \Leftrightarrow \; p=-\frac{2}{3}$$

ii) The plane are parallel when the normal vectors are linearly dependent: $\mathbf{n}=k\mathbf{n}'$

Considering the "z" component we need to have $k=2$.

$$and \quad 2(4\text{p}+1)=6 \;\text{ and }\; 2(p-2)=-3$$

$$p=\frac{1}{2} \quad and \quad p=\frac{1}{2}$$

b) $p=4$

$$i) \begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}6\\-3\\2\end{pmatrix}=42 \Leftrightarrow 6x-3y+2z=42$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}17\\2\\1\end{pmatrix}=-7 \Leftrightarrow 17x+2y+z=-7$$

ii) To find an equation of the line, we need a point and a vector.

Let solve these equations simultaneously by choosing $x=0$

$$Thisgives: \begin{cases}-3y+2z=42 & l_1\\ 2y+z=-7 & l_2\end{cases}$$

$$l_1-2l_2 \;\; gives -7y=56 \;\; or \;\; y=-8$$

$$2l_1+3l_2 \;\; gives \;\; 7z=63 \;\; or \;\; z=9$$

The point (0,-8,9) belong to the line.

A direction vector to the line is $\begin{pmatrix}6\\-3\\2\end{pmatrix}\times\begin{pmatrix}17\\2\\1\end{pmatrix}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 6 & -3 & 2\\ 17 & 2 & 1\end{vmatrix}=-7\mathbf{i}+28\mathbf{j}+63\mathbf{k}$

Another direction vector is $\mathbf{i}-4\mathbf{j}-9\mathbf{k}$

An equation of the line of intersection is $\mathbf{r}=\begin{pmatrix}0\\-8\\9\end{pmatrix}+\lambda\begin{pmatrix}-1\\4\\9\end{pmatrix}$

c) A direction vector of the plane is $-\mathbf{i}+4\mathbf{j}+9\mathbf{k}$

Another one is $\begin{pmatrix}30-0\\7+8\\30-9\end{pmatrix}=30\mathbf{i}+15\mathbf{j}+21\mathbf{k}$

An equation of the plane is $\mathbf{r}=\begin{pmatrix}0\\-8\\9\end{pmatrix}+\lambda\begin{pmatrix}-1\\4\\9\end{pmatrix}+\mu\begin{pmatrix}30\\15\\21\end{pmatrix}$ *or*

$$\mathbf{r}=\begin{pmatrix}0\\-8\\9\end{pmatrix}+\lambda\begin{pmatrix}-1\\4\\9\end{pmatrix}+\mu\begin{pmatrix}10\\5\\7\end{pmatrix}$$

Exam report

Question 6 and 7

The structure of these questions proved very helpful to candidates, and those who were reasonably careful could generally score at least 13 of question 6's 16 marks and 16 of question 7's 18 marks without any difficulty. The main hurdle arose in question 6, where many candidates failed to realise in part (b) that they had already worked out the eigenvalues of **M** in part (a), and thus they went ahead and started all over again in an effort to find, and then solve, a cubic characteristic equation. This was all a bit of a waste of time and frequently not as successful as their former effort.

Question 7:	Exam report

$$a)\ \begin{vmatrix}16-q & 5 & 7\\ -12 & -1-q & -7\\ 6 & 6 & 10-q\end{vmatrix}\begin{matrix}l_1\\ l_2\\ l_3\end{matrix} = \begin{vmatrix}4-q & 4-q & 0\\ -12 & -1-q & -7\\ 6 & 6 & 10-q\end{vmatrix}\begin{matrix}l_1+l_2\\ l_2\\ l_3\end{matrix} = (4-q)\begin{vmatrix}1 & 1 & 0\\ -12 & -1-q & -7\\ 6 & 6 & 10-q\end{vmatrix}$$

$$ii)\ \Delta = (4-q)\begin{vmatrix}0 & 1 & 0\\ q-11 & -1-q & -7\\ 0 & 6 & 10-q\end{vmatrix} = (4-q)(q-11)(q-10)$$

$$b)\ i)\ \mathbf{M}\times\begin{pmatrix}2\\ 5\\ -7\end{pmatrix} = \begin{bmatrix}16 & 5 & 7\\ -12 & -1 & -7\\ 6 & 6 & 10\end{bmatrix}\times\begin{pmatrix}2\\ 5\\ -7\end{pmatrix} = \begin{pmatrix}32+25-49\\ -24-5+49\\ 12+30-70\end{pmatrix} = \begin{pmatrix}8\\ 20\\ -28\end{pmatrix} = 4\begin{pmatrix}2\\ 5\\ -7\end{pmatrix}$$

$ii)$ The other eigenvalues of $\mathbf{M}$ are 11 and 10

For $\lambda=11$, let solve $\mathbf{M}\times\begin{pmatrix}x\\ y\\ z\end{pmatrix} = 11\begin{pmatrix}x\\ y\\ z\end{pmatrix}$ *or* $(\mathbf{M}-11\mathbf{I})\times\begin{pmatrix}x\\ y\\ z\end{pmatrix} = 0$

This gives $\begin{cases}5x+5y+7z=0\\ -12x-12y-7z=0\\ 6x+6y-z=0\end{cases}$ *and an "obvious" solution is* $\begin{pmatrix}1\\ -1\\ 0\end{pmatrix}$

For $\lambda=10$, let solve $\mathbf{M}\times\begin{pmatrix}x\\ y\\ z\end{pmatrix} = 10\begin{pmatrix}x\\ y\\ z\end{pmatrix}$ *or* $(\mathbf{M}-10\mathbf{I})\times\begin{pmatrix}x\\ y\\ z\end{pmatrix} = 0$

This gives $\begin{cases}6x+5y+7z=0\\ -12x-11y-7z=0\\ 6x+6y=0\end{cases}$ $\begin{cases}x=-y \quad \text{from } l_3\\ 7z=y \quad \text{from } l_1\\ \text{Let's choose } z=1\end{cases}$ *this gives* $\begin{pmatrix}-7\\ 7\\ 1\end{pmatrix}$

$c)$ Any line with an eigenvector as direction vector and going through O is invariant.

for example : $\frac{x}{2}=\frac{y}{5}=\frac{z}{-7}$ *or* $(x=-y \text{ and } z=0)$ *or* $\frac{x}{-7}=\frac{y}{7}=z.$

Grade boundaries

Grade			A	B	C	D	E
Mark	Max 75		63	55	47	39	32

MY SCORE FOR THIS PAPER IS ..... / 75

MY GRADE : ....