General Certificate of Education
January 2009
Advanced Level Examination

MATHEMATICS **MFP4**
Unit Further Pure 4

Tuesday 27 January 2009 1.30 pm to 3.00 pm

For this paper you must have:
- a 12-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP4.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 The line l has equation $\mathbf{r} = (1 + 4t)\mathbf{i} + (-2 + 12t)\mathbf{j} + (1 - 3t)\mathbf{k}$.

(a) Write down a direction vector for l. *(1 mark)*

(b) (i) Find direction cosines for l. *(2 marks)*

(ii) Explain the geometrical significance of the direction cosines in relation to l. *(1 mark)*

(c) Write down a vector equation for l in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$. *(2 marks)*

2 The 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ are such that

$$\mathbf{AB} = \begin{bmatrix} 9 & 1 \\ 7 & 13 \end{bmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{bmatrix} 14 & 2 \\ 1 & 8 \end{bmatrix}$$

Without finding $\mathbf{A}$ and $\mathbf{B}$:

(a) find the value of $\det \mathbf{B}$, given that $\det \mathbf{A} = 10$; *(3 marks)*

(b) determine the 2×2 matrices $\mathbf{C}$ and $\mathbf{D}$ given by

$$\mathbf{C} = (\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}) \quad \text{and} \quad \mathbf{D} = (\mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}})^{\mathrm{T}}$$

where $\mathbf{M}^{\mathrm{T}}$ denotes the transpose of matrix $\mathbf{M}$. *(3 marks)*

3 The points X, Y and Z have position vectors

$$\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{z} = \begin{bmatrix} -8 \\ 1 \\ a \end{bmatrix}$$

respectively, relative to the origin O.

(a) Find:

(i) $\mathbf{x} \times \mathbf{y}$; *(2 marks)*

(ii) $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}$. *(2 marks)*

(b) Using these results, or otherwise, find:

(i) the area of triangle OXY; *(2 marks)*

(ii) the value of a for which $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are linearly dependent. *(2 marks)*

4 (a) Given that -1 is an eigenvalue of the matrix $\mathbf{M} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}$, find a corresponding eigenvector. *(3 marks)*

(b) Determine the other two eigenvalues of $\mathbf{M}$, expressing each answer in its simplest surd form. *(8 marks)*

5 (a) Expand the determinant

$$D = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ z & x & y \end{vmatrix}$$ *(2 marks)*

(b) Show that $(x + y + z)$ is a factor of the determinant

$$\Delta = \begin{vmatrix} x & y & z \\ y - z & z - x & x - y \\ x + z & y + x & z + y \end{vmatrix}$$ *(2 marks)*

(c) Show that $\Delta = k(x + y + z)D$ for some integer k. *(3 marks)*

Turn over ▶

6 The line L and the plane Π are, respectively, given by the equations

$$\mathbf{r} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{r} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 20$$

(a) Determine the size of the acute angle between L and Π. *(4 marks)*

(b) The point P has coordinates $(10, -5, 37)$.

(i) Show that P lies on L. *(1 mark)*

(ii) Find the coordinates of the point Q where L meets Π. *(4 marks)*

(iii) Deduce the distance PQ and the shortest distance from P to Π. *(3 marks)*

7 Two fixed planes have equations

$$\begin{aligned} x - 2y + z &= -1 \\ -x + y + 3z &= 3 \end{aligned}$$

(a) The point P, whose z-coordinate is λ, lies on the line of intersection of these two planes. Find the x- and y-coordinates of P in terms of λ. *(3 marks)*

(b) The point P also lies on the variable plane with equation $5x + ky + 17z = 1$. Show that

$$(k + 13)(2\lambda - 1) = 0$$ *(3 marks)*

(c) For the system of equations

$$\begin{aligned} x - 2y + z &= -1 \\ -x + y + 3z &= 3 \\ 5x + ky + 17z &= 1 \end{aligned}$$

determine the solution(s), if any, of the system, and their geometrical significance in relation to the three planes, in the cases:

(i) $k = -13$;

(ii) $k \neq -13$. *(6 marks)*

8 The plane transformation T has matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, and maps points (x, y) onto image points (X, Y) such that

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

(a) (i) Find $\mathbf{A}^{-1}$. *(2 marks)*

(ii) Hence express each of x and y in terms of X and Y. *(2 marks)*

(b) Give a full geometrical description of T. *(5 marks)*

(c) Any plane curve with equation of the form $\dfrac{x^2}{p} + \dfrac{y^2}{q} = 1$, where p and q are distinct positive constants, is an ellipse.

(i) Show that the curve E with equation $6x^2 + y^2 = 3$ is an ellipse. *(1 mark)*

(ii) Deduce that the image of the curve E under T has equation

$$2X^2 + 4XY + 5Y^2 = 15$$

(2 marks)

(iii) Explain why the curve with equation $2x^2 + 4xy + 5y^2 = 15$ is an ellipse. *(1 mark)*

END OF QUESTIONS

Key to mark scheme and abbreviations used in marking

M	mark is for method
m or dM	mark is dependent on one or more M marks and is for method
A	mark is dependent on M or m marks and is for accuracy
B	mark is independent of M or m marks and is for method and accuracy
E	mark is for explanation

√or ft or F	follow through from previous incorrect result	MC	mis-copy
CAO	correct answer only	MR	mis-read
CSO	correct solution only	RA	required accuracy
AWFW	anything which falls within	FW	further work
AWRT	anything which rounds to	ISW	ignore subsequent work
ACF	any correct form	FIW	from incorrect work
AG	answer given	BOD	given benefit of doubt
SC	special case	WR	work replaced by candidate
OE	or equivalent	FB	formulae book
A2,1	2 or 1 (or 0) accuracy marks	NOS	not on scheme
$-x$ EE	deduct x marks for each error	G	graph
NMS	no method shown	c	candidate
PI	possibly implied	sf	significant figure(s)
SCA	substantially correct approach	dp	decimal place(s)

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded. However, there are situations in some units where part marks would be appropriate, particularly when similar techniques are involved. Your Principal Examiner will alert you to these and details will be provided on the mark scheme.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

MFP4

Q	Solution	Marks	Total	Comments
1(a)	$4\mathbf{i} + 12\mathbf{j} - 3\mathbf{k}$ or equivalent	B1	1	
(b)(i)	$\sqrt{4^2 + 12^2 + 3^2} = 13$	M1		ft From their d.v.
	d.c.'s are $\frac{4}{13}, \frac{12}{13}$ and $-\frac{3}{13}$	A1F	2	ft
(ii)	The cosines of the angles between the line and the coordinate axes	B1	1	
(c)	$\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b}$ = their d.v.	B1 B1F	 2	CAO ft
	Total		**6**	
2(a)	det $\mathbf{AB}$ = 110 Use of det $\mathbf{AB}$ = det $\mathbf{A}$ det $\mathbf{B}$ det $\mathbf{B}$ = 11	B1 M1 A1F	 3	 ft their det AB / 10
(b)	$\mathbf{C} = (\mathbf{AB})^T = \begin{bmatrix} 9 & 7 \\ 1 & 13 \end{bmatrix}$	M1 A1		
	$\mathbf{D} = [\ (\mathbf{BA})^T\ ]^T = \mathbf{BA} = \begin{bmatrix} 14 & 2 \\ 1 & 8 \end{bmatrix}$	B1	3	For reference: $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 5 & 3 \\ -2 & 1 \end{bmatrix}$
	Total		**6**	
3(a)(i)	$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 2 \\ 5 & 7 & 4 \end{vmatrix} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$	M1 A1	 2	
(ii)	$(\mathbf{x} \times \mathbf{y}) \bullet \mathbf{z} = \begin{vmatrix} 2 & 3 & 2 \\ 5 & 7 & 4 \\ -8 & 1 & a \end{vmatrix}$	M1		or via $\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} -8 \\ 1 \\ a \end{bmatrix}$
	$= 18 - a$	A1F	2	ft
(b)(i)	$A = ½\ \lvert \mathbf{x} \times \mathbf{y} \rvert$ $= ½\sqrt{2^2 + 2^2 + 1^2} = 1.5$	M1 A1F	 2	 ft
(ii)	$(\mathbf{x} \times \mathbf{y}) \bullet \mathbf{z} = 0 \Rightarrow a = 18$	M1 A1F	 2	 ft or CAO from new start
	Total		**8**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
4(a)	Subst[g]. $\lambda = -1$ into $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$	M1		Or $\mathbf{M}\,\mathbf{x} = -\,\mathbf{x}$ etc.
	Solving between $x + y + z = 0$ and $x + y + 2z = 0$	dM1		
	Eigenvector(s) $\alpha\begin{bmatrix}1\\-1\\0\end{bmatrix}$	A1	3	Any non-zero α will suffice
(b)	Attempt at Char. Eqn.	M1		
	$\lambda^3 - 5\lambda^2 - 5\lambda + 1 = 0$	A1 × 3		Each coefft. (not the λ^3)
	Use of division/factor theorem etc.	M1		With/without $(\lambda + 1)$ factor
	$(\lambda + 1)(\lambda^2 - 6\lambda + 1)$	A1		
	Solving remaining quadratic factor	M1		
	$\lambda_{2,3} = 3 \pm 2\sqrt{2}$	A1	8	CAO simplest exact form
	Total		**11**	
5(a)	$D = x^2 + y^2 + z^2 - xy - yz - zx$	M1 A1	2	
(b)	E.g. by $C_1' = C_1 + (C_2 + C_3)$	M1		
	$\Rightarrow \Delta = \begin{vmatrix}x+y+z & y & z\\0 & z-x & x-y\\2(x+y+z) & y+x & z+y\end{vmatrix}$			
	$= (x + y + z)\begin{vmatrix}1 & y & z\\0 & z-x & x-y\\2 & y+x & z+y\end{vmatrix}$	A1	2	Shown or explained from previous line
(c)	Working on (R/C-ops) or expanding remaining determinant	M1		
	2nd factor $= -(x^2 + y^2 + z^2 - xy - yz - zx)$	dM1		Good attempt
	$k = -1$	A1	3	
	Total		**7**	
6(a)	Use of $\sin\theta$ or $\cos\theta$ = (dot product)/(product of moduli)	M1		Must be d.v. of line & plane's nml.
	Num[r]. = 3	B1F		ft
	Denom[r]. = $\sqrt{18}\sqrt{2}$	B1F		ft
	$\theta = 30^\circ$	A1	4	CAO
(b)(i)	$\lambda = 8$ noted or found	B1	1	
(ii)	$\begin{bmatrix}2+\lambda\\3-\lambda\\5+4\lambda\end{bmatrix}\bullet\begin{bmatrix}0\\1\\1\end{bmatrix} = 20$	M1		Attempt at this
	$3 - \lambda + 5 + 4\lambda = 20 \Rightarrow \lambda = 4$	M1 A1		Solving a linear eqn. in λ
	giving $Q = (6, -1, 21)$	B1F	4	ft
(iii)	$PQ = \sqrt{4^2 + 4^2 + 16^2} = 12\sqrt{2}$	M1 A1		Or $4\sqrt{18}$, 17.0, 16.97 etc.
	Sh. Dist. = $12\sqrt{2}\,.\sin 30^\circ = 6\sqrt{2}$	B1F	3	ft $\frac{1}{2}$ previous answer
	Total		**12**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
7(a)	$x - 2y = -1 - \lambda$			
	$-x + y = 3 - 3\lambda$	B1		At least one correct from setting $z = \lambda$
	Solving for x and y in terms of λ	M1		
	$x = 7\lambda - 5$ and $y = 4\lambda - 2$	A1	3	CAO
(b)	Subst^g. x, y, z in terms of λ in $5x + ky + 17z = 1$	M1		
	$35\lambda - 25 + k(4y - 2) + 17\lambda - 1 = 0$			
	Factsn. attempt: $(4y - 2)(k + 13) = 0$	dM1		
	$(2y - 1)(k + 13) = 0$	A1	3	**ANSWER GIVEN**
(c)(i)	When $k = -13$, $5x - 13y + 17z$			
	$= 35\lambda - 25 - 52\lambda + 26 + 17\lambda \equiv 1$	B1		Subst^g. into 3rd eqn. and demonstrating consistency
	The three planes intersect in a line	B1		
	Solns. $x = 7\lambda - 5$, $y = 4\lambda - 2$, $z = \lambda$	B1F		ft
(ii)	When $k \neq -13$, $\lambda = ½$	B1		
	Soln. $(-1½, 0, ½)$	B1F		ft
	Three planes meet at a point	B1	6	
	Total		**12**	
8(a)(i)	$\frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$	B1 B1	2	1/det Transposed matx. of cofactors
(ii)	$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \frac{1}{5}(X+2Y) \\ \frac{1}{5}(Y-2X) \end{bmatrix}$	M1 A1F	2	ft
(b)	$\mathbf{A} = \sqrt{5}\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$	B1		
	Enlargement sf $\sqrt{5}$ (centre O)	M1 A1		Two components in any order
	+ Rotation thro' $\cos^{-1}(1/\sqrt{5})$	M1 A1	5	or 63.4° or 1.11 rads
(c)(i)	$p = ½$, $q = 3$ noted	B1	1	Or form $\frac{x^2}{\frac{1}{2}} + \frac{y^2}{3} = 1$ shown
(ii)	$6x^2 + y^2 = 3 \Rightarrow$			
	$\frac{6}{25}\left(X^2 + 4XY + 4Y^2\right) + \frac{1}{25}\left(Y^2 - 4XY + 4X^2\right) = 3$	M1		Subst^g. for x and y
	$\Rightarrow 10X^2 + 20XY + 25Y^2 = 75$			
	$\Rightarrow 2X^2 + 4XY + 5Y^2 = 15$	A1	2	**ANSWER GIVEN**
(iii)	It is just an enlarged rotation of E, hence still an ellipse	B1	1	
	Total		**13**	
	TOTAL		**75**	

AQA – Further pure 4 – Jan 2009 – Answers

Question 1:	Exam report
$a)$ *direction vector* $\mathbf{u} = \begin{pmatrix} 4 \\ 12 \\ -3 \end{pmatrix}$ (*coefficients of the parameter* t) $b\,i)\ \lvert\mathbf{u}\rvert = \sqrt{4^2 + 12^2 + (-3)^2} = 13$ The direction cosines are : $\frac{4}{13}, \frac{12}{13}, \frac{-3}{13}$ $ii)$ The direction cosines are the cosine of the angles made by the line and the axes. $c)$ The point $A(1,-2,1)$ belongs to the line and $\mathbf{u} = 4\mathbf{i} + 12\mathbf{j} - 3\mathbf{k}$ is a direction vector An equation of the line is $\left(\mathbf{r} - \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right) \times \begin{pmatrix} 4 \\ 12 \\ -3 \end{pmatrix} = 0$	This was a straightforward starter to the paper, and was generally found to be so by candidates. Some candidates did not seem to know what direction cosines are, though, and there were others who apparently had not seen the required form for the equation of a line in part (c).

Question 2:	Exam report
$a)\ \det(\mathbf{AB}) = \det(\mathbf{A}) \times \det(\mathbf{B})$ $\det(\mathbf{AB}) = \begin{vmatrix} 9 & 1 \\ 7 & 13 \end{vmatrix} = 9 \times 13 - 7 \times 1 = 110$ *so* $110 = 10 \times \det(\mathbf{B})$ *this gives* $\det(\mathbf{B}) = 11$ $b)\ \mathbf{C} = \left(\mathbf{B}^T\mathbf{A}^T\right) = \left(\mathbf{AB}\right)^T = \begin{pmatrix} 9 & 7 \\ 1 & 13 \end{pmatrix}$ $\mathbf{D} = \left(\mathbf{A}^T\mathbf{B}^T\right)^T = (\mathbf{B}^T)^T(\mathbf{A}^T)^T = \mathbf{BA} = \begin{pmatrix} 14 & 2 \\ 1 & 8 \end{pmatrix}$	This question was intended to test a couple of basic areas of the module's work on determinants and transposes. The former topic was generally handled very easily, but a majority of candidates appeared to resort to guessing what was involved in part (b) with the work on transposes. In general, candidates scored either 3 or 6 marks on this question.

Question 3:	Exam report
$a)\,i)\ \mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 2 \\ 5 & 7 & 4 \end{vmatrix} = (12 - 14)\mathbf{i} - (8 - 10)\mathbf{j} + (14 - 15)\mathbf{k}$ $= -2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ $ii)\ (\mathbf{x} \times \mathbf{y}).\mathbf{z} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}.\begin{pmatrix} -8 \\ 1 \\ a \end{pmatrix} = 16 + 2 - a = 18 - a$ $b)\,i)$ Area of OXY $= \frac{1}{2}\lvert\mathbf{x} \times \mathbf{y}\rvert = \frac{1}{2}\sqrt{(-2)^2 + 2^2 + (-1)^2} = \frac{3}{2}$ $ii)\ \mathbf{x}, \mathbf{y}, \mathbf{z}$ are linearly dependent when $(\mathbf{x} \times \mathbf{y}).\mathbf{z} = 0$ This occurs when $a = 18$	This was undoubtedly the easiest question on the paper, with a majority of candidates scoring all 8 marks on it. Even those making a mistake early on generally only lost the one mark because of the follow-through permitted.

Question 4:

$a)$ Let's solve the equation $(\mathbf{M}+1\mathbf{I})\begin{pmatrix}x\\y\\z\end{pmatrix}=0$

$$\begin{pmatrix}2&2&2\\2&2&2\\2&2&4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=0 \Leftrightarrow \begin{cases}2x+2y+2z=0\\2x+2y+2z=0\\2x+2y+4z=0\end{cases} \Leftrightarrow \begin{cases}x+y+z=0\\x+y+2z=0\end{cases}$$

An eigenvector is $\begin{pmatrix}1\\-1\\0\end{pmatrix}$

$b)$ To find the eigenvalues, we solve $\det(\mathbf{M}-\lambda\mathbf{I})=0$.

$$\begin{vmatrix}1-\lambda&2&2\\2&1-\lambda&2\\2&2&3-\lambda\end{vmatrix}=(1-\lambda)\big((1-\lambda)(3-\lambda)-4\big)-2\big(2(3-\lambda)-4\big)+2\big(4-2(1-\lambda)\big)$$

$$\det(\mathbf{M}-\lambda\mathbf{I})=(1-\lambda)\left(\lambda^2-4\lambda-1\right)-2(2-2\lambda)+2(2+2\lambda)$$

$$=-\lambda^3+5\lambda^2-3\lambda-1+8\lambda=-\lambda^3+5\lambda^2+5\lambda-1$$

$\det(\mathbf{M}-\lambda\mathbf{I})=0 \Leftrightarrow \lambda^3-5\lambda^2-5\lambda+1=0$

We know that -1 is a root so we can factorise by $(\lambda+1)$

$$(\lambda+1)(\lambda^2-6\lambda+1)=0$$

The discriminant of the quadratic equation is $(-6)^2-4\times1\times1=32$

the roots are $\lambda_2=\dfrac{6+\sqrt{32}}{2}=3+2\sqrt{2}$ *and* $\lambda_3=3-2\sqrt{2}$

Exam report

The purpose of part (a) was to give candidates a linear factor of the cubic characteristic equation that was to arise in part (b). Most took the hint but then spent far too much time in obtaining this cubic, extracting the linear factor correctly and then solving the resulting quadratic equation. Even amongst those who reached the end successfully, the final 'simplest surd form' requirement for the answer caught far too many out.

Question 5:

$$D=\begin{vmatrix}1&1&1\\x&y&z\\z&x&y\end{vmatrix}=1\begin{vmatrix}y&z\\x&y\end{vmatrix}-1\begin{vmatrix}x&z\\z&y\end{vmatrix}+1\begin{vmatrix}x&y\\z&x\end{vmatrix}=y^2-xz-xz+z^2+x^2-zy$$

$D=x^2+y^2+z^2-(xz+xy+yz)$

$b)$ By adding ther three columns of the determinant we obtain

$$\Delta=\begin{vmatrix}x+y+z&y&z\\0&z-x&x-y\\2x+2y+2z&y+x&z+y\end{vmatrix}=(x+y+z)\begin{vmatrix}1&y&z\\0&z-x&x-y\\2&y+x&z+y\end{vmatrix}$$

$$c)\ \Delta=(x+y+z)\times1\begin{vmatrix}z-x&x-y\\y+x&z+y\end{vmatrix}$$

$$\Delta=(x+y+z)\Big((z-x)(z+y)-(x+y)(x-y)+2\big(y(x-y)-z(z-x)\big)\Big)$$

$$\Delta=(x+y+z)\left(z^2+zy-zx-xy-y^2-x^2+2xy-2y^2-2z^2+2zx\right)$$

$$\Delta=(x+y+z)\left(-x^2-y^2-z^2+xy+zx+zy\right)=-(x+y+z)D \qquad k=-1$$

Exam report

Throughout all the MFP4 papers set thus far, this topic of determinants has proved to be a problem to many candidates. It is clear that while some candidates *have* seen how to perform row and column operations on determinants, others equally clearly have not.

The phrase "expand the determinant" in part (a) was supposed to direct candidates away from using row and/or column operations, but many did not take the hint. Conversely, the set-up in part (b) obviously requires row/column operations to begin with but, once a linear factor has been extracted (the one given), candidates should have been able to spot that they can then expand directly, as they only have a quadratic expression to deal with. Some candidates went on to use row/column operations very concisely and successfully, but they were very much the exception rather than the rule.

Question 6:	Exam report
$a)$ A direction vector of the line L is $\mathbf{u}=\begin{pmatrix}1\\-1\\4\end{pmatrix}$ and a normal vector to the plane is $\mathbf{n}=\begin{pmatrix}0\\1\\1\end{pmatrix}$ Let's work out the angle between these two vectors $\mathbf{u.n}=\begin{pmatrix}1\\-1\\4\end{pmatrix}.\begin{pmatrix}0\\1\\1\end{pmatrix}=0-1+4=3$ $\lvert\mathbf{u}\rvert=\sqrt{1^2+(-1)^2+4^2}=\sqrt{18}=3\sqrt{2}$ $\lvert\mathbf{n}\rvert=\sqrt{0^2+1^2+1^2}=\sqrt{2}$ $Cos\theta=\frac{\mathbf{u.n}}{\lvert\mathbf{u}\rvert\lvert\mathbf{n}\rvert}=\frac{3}{3\sqrt{2}\times\sqrt{2}}=\frac{1}{2}$ The angle between the vector is 60^o The acute angle between the line L and the plane is $90^0-60^o=30^o$. $b)\begin{pmatrix}10\\-5\\37\end{pmatrix}-\begin{pmatrix}2\\3\\5\end{pmatrix}=\begin{pmatrix}8\\-8\\32\end{pmatrix}=8\begin{pmatrix}1\\-1\\4\end{pmatrix}$ P belongs to L (λ=8) $ii)$ Let's write the position vector of Q : $\mathbf{q}=\begin{pmatrix}2+\lambda\\3-\lambda\\5+4\lambda\end{pmatrix}$ Q belongs to the plane so $\mathbf{q}.\begin{pmatrix}0\\1\\1\end{pmatrix}=20 \Leftrightarrow \begin{pmatrix}2+\lambda\\3-\lambda\\5+4\lambda\end{pmatrix}.\begin{pmatrix}0\\1\\1\end{pmatrix}=20$ This gives the equation: $3-\lambda+5+4\lambda=20$ so $\lambda=4$ and $\mathbf{q}=\begin{pmatrix}2+4\\3-4\\5+4\times4\end{pmatrix}=\begin{pmatrix}6\\-1\\21\end{pmatrix}$ $iii)$ The vector $\overrightarrow{PQ}=\mathbf{q}-\mathbf{p}=\begin{pmatrix}-4\\-4\\16\end{pmatrix}$ $and\ PQ=\lvert\overrightarrow{PQ}\rvert=\sqrt{(-4)^2+(-4)^2+16^2}=12\sqrt{2}$ If we name H the point on the plane so that PH is the shortest distance from P to Π then the traingle PQH is a right angle angle and in this triangle PH=PQ$\times$Sin30^o $PH=12\sqrt{2}\times\frac{1}{2}=6\sqrt{2}$	In part (a), the complementary angle to the one asked for was often found. In part (c)(iii), most candidates found a distance *PQ*, although rather too many just found a vector $\overrightarrow{PQ}$ Slightly surprisingly, most candidates seemed to think that this was the shortest distance required.

Question 7:	Exam report
$a)$ If $z=\lambda$, the equations becomes $\begin{cases} x-2y=-1-\lambda \\ -x+y=3-3\lambda \end{cases}$ *by adding the equations we have* $-y=2-4\lambda$; $y=-2+4\lambda$ *and* $-x+y=3-3\lambda$ *becomes* $-x-2+4\lambda=3-3\lambda$ $x=-5+7\lambda$ *Conclusion* : $\mathbf{p}=\begin{pmatrix} -5+7\lambda \\ -2+4\lambda \\ \lambda \end{pmatrix}$ $b)$ P lies on the plane with equation $5x+ky+17z=1$ by substitution, we have $5(-5+7\lambda)+k(-2+4\lambda)+17\lambda=1$ $-25+35\lambda-2k+4k\lambda+17\lambda=1$ $32\lambda+4k\lambda-2k-26=0$ $16\lambda+2k\lambda \quad -k-13=0$ $2\lambda(13+k) \quad -1(13+k)=0$ $(13+k)(2\lambda-1)=0$ $c)$ • $i)$ *if* $k=-13$, *then* $(13+k)(2\lambda-1)=0$ for all values of λ. This means that the line of intersection is included in the plane i.e the three planes intersect in a line. The solution is the line with equation $\mathbf{r}=\begin{pmatrix} -5 \\ -2 \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}$ • $ii)$ *if* $k\neq-13$, *then* $(13+k)(2\lambda-1)=0$ *only for* $\lambda=\frac{1}{2}$ The three planes intersect at a unique point $P\left(-5+\frac{7}{2}, -2+\frac{4}{2}, 0+\frac{1}{2}\right)$ $P\left(-\frac{3}{2},0,\frac{1}{2}\right)$	This question was definitely the most demanding on the paper; partly because this topic of solving systems of equations has generally proved to be one of the least palatable topics to candidates over the years, but mostly due to the poor examination technique employed on this occasion. Many candidates ignored the directly-stated demand to set $z = \lambda$ in part (a). It was unfortunate that few candidates seemed to notice that the given result in part (b) should be used in part (c). This resulted in much unnecessary work for many candidates and often a failure to answer the question asked. The demand was for candidates to "determine" any solutions, and then to describe what they were in relation to the planes. Many candidates ignored the first of these requests altogether, and then went on to describe the arrangement of the planes, which was not what the question said. A large majority of candidates lost four or five marks, in part (c)(i) by failing to re-state their answer to part (a) and then calling it a "line" (of intersection) and in part (c)(ii) by failing to notice the consequence of part (b)'s answer that $z = \lambda = ½$, which then gives a point (of intersection) whose coordinates can be written straight down (again, using part (a)'s answer).

Question 8:

$a)\,i)\,\det(\mathbf{A})=\begin{vmatrix}1 & -2\\ 2 & 1\end{vmatrix}=1+4=5$

$$\mathbf{A}^{-1}=\frac{1}{5}\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}$$

$ii)\,\mathbf{A}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}X\\ Y\end{pmatrix}$ so $\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix}x\\ y\end{pmatrix}=\mathbf{A}^{-1}\begin{pmatrix}X\\ Y\end{pmatrix}$

$$\begin{pmatrix}x\\ y\end{pmatrix}=\mathbf{A}^{-1}\begin{pmatrix}X\\ Y\end{pmatrix}$$

This gives $\begin{cases}x=\frac{1}{5}(X+2Y)\\ y=\frac{1}{5}(-2X+Y)\end{cases}$

$b)\,\mathbf{A}=\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}=\sqrt{5}\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}=\sqrt{5}\begin{pmatrix}Cos\theta & -Sin\theta\\ Sin\theta & Cos\theta\end{pmatrix}$

with $\theta=Cos^{-1}(\frac{1}{\sqrt{5}})=63.4^o$ to 1 d.p

The tranformation T is an enlargement by scale $\sqrt{5}$ *and* a rotation anticlockwise angle θ.

$c)\,i)\,6x^2+y^2=3$

$$2x^2+\frac{y^2}{3}=1 \qquad \frac{x^2}{\frac{1}{2}}+\frac{y^2}{3}=1 \text{ is an ellipse}$$

$ii)$ substituting x by $\frac{1}{5}(X+2Y)$ and y by $\frac{1}{5}(-2X+Y)$

The equation of the ellipse, $6x^2+y^2=3$, becomes

$$6\left(\frac{1}{5}(X+2Y)\right)^2+\left(\frac{1}{5}(-2X+Y)\right)^2=3 \quad (\times 25)$$

$$6(X+2Y)^2+(-2X+Y)^2=75$$

$$6X^2+24Y^2+24XY+4X^2+Y^2-4XY=75$$

$$10X^2+25Y^2+20XY=75 \qquad (/5)$$

$$2X^2+5Y^2+4XY=15$$

$iii)$ Through an enlargement and a rotation, the image of an ellipse remains an ellipse.

Exam report

It was disappointing to see so many candidates unable to find correctly the inverse of a 2×2 matrix at the outset of this question. The description in part (b) proved tricky — as had been expected — but actually only relies on understanding of the FP1 work on matrices and transformations. Taking this into account, it was sad to see so few attempts to describe T.

In part (c)(i), many candidates failed to point out the values of *p* and *q* that would justify *E*'s status as an ellipse. In most cases, this arose as a result of an inability to divide correctly throughout the given equation by 3. By the time it came to the final two parts, most candidates had made at least one mistake which deprived them of the chance of further successful progress.

MY SCORE FOR THIS PAPER IS ..... / 75

MY GRADE : ....

Grade boundaries

Grade			A	B	C	D	E
Mark	Max 75		56	49	42	35	28