General Certificate of Education
January 2008
Advanced Level Examination

MATHEMATICS **MFP4**
Unit Further Pure 4

Wednesday 30 January 2008 9.00 am to 10.30 am

For this paper you must have:
• a 12-page answer book
• the blue AQA booklet of formulae and statistical tables.
You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions

- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP4.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information

- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice

- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 Give a full geometrical description of the transformation represented by each of the following matrices:

(a) $\begin{bmatrix} 0.8 & 0 & -0.6 \\ 0 & 1 & 0 \\ 0.6 & 0 & 0.8 \end{bmatrix}$; *(3 marks)*

(b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. *(2 marks)*

2 It is given that $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$ and $\mathbf{c} = \mathbf{i} + 4\mathbf{j} + 28\mathbf{k}$.

(a) Determine:

(i) $\mathbf{a} \cdot \mathbf{b}$; *(1 mark)*

(ii) $\mathbf{a} \times \mathbf{b}$; *(2 marks)*

(iii) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. *(2 marks)*

(b) Describe the geometrical relationship between the vectors:

(i) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$; *(1 mark)*

(ii) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. *(1 mark)*

3 A shear S is represented by the matrix $\mathbf{A} = \begin{bmatrix} p & q \\ -q & r \end{bmatrix}$, where p, q and r are constants.

(a) By considering one of the geometrical properties of a shear, explain why $pr + q^2 = 1$. *(2 marks)*

(b) Given that $p = 4$ and that the image of the point $(-1, 2)$ under S is $(2, -1)$, find:

(i) the value of q and the value of r; *(3 marks)*

(ii) the equation of the line of invariant points of S. *(3 marks)*

4 The matrix $\mathbf{T}$ has eigenvalues 2 and -2, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ respectively.

(a) Given that $\mathbf{T} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}$, where $\mathbf{D}$ is a diagonal matrix, write down suitable matrices $\mathbf{U}$, $\mathbf{D}$ and $\mathbf{U}^{-1}$. *(3 marks)*

(b) Hence prove that, for all **even** positive integers n,

$$\mathbf{T}^n = \mathrm{f}(n)\,\mathbf{I}$$

where $\mathrm{f}(n)$ is a function of n, and $\mathbf{I}$ is the 2×2 identity matrix. *(5 marks)*

5 A system of equations is given by

$$\begin{aligned} x + 3y + 5z &= -2 \\ 3x - 4y + 2z &= 7 \\ ax + 11y + 13z &= b \end{aligned}$$

where a and b are constants.

(a) Find the unique solution of the system in the case when $a = 3$ and $b = 2$. *(5 marks)*

(b) (i) Determine the value of a for which the system does not have a unique solution. *(3 marks)*

(ii) For this value of a, find the value of b such that the system of equations is consistent. *(4 marks)*

Turn over for the next question

6 (a) The line l has equation $\mathbf{r} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$.

(i) Write down a vector equation for l in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$. *(1 mark)*

(ii) Write down cartesian equations for l. *(2 marks)*

(iii) Find the direction cosines of l and explain, geometrically, what these represent. *(3 marks)*

(b) The plane Π has equation $\mathbf{r} = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

(i) Find an equation for Π in the form $\mathbf{r} \cdot \mathbf{n} = d$. *(4 marks)*

(ii) State the geometrical significance of the value of d in this case. *(1 mark)*

(c) Determine, to the nearest 0.1°, the angle between l and Π. *(4 marks)*

7 The non-singular matrix $\mathbf{M} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

(a) (i) Show that

$$\mathbf{M}^2 + 2\mathbf{I} = k\mathbf{M}$$

for some integer k to be determined. *(3 marks)*

(ii) By multiplying the equation in part (a)(i) by $\mathbf{M}^{-1}$, show that

$$\mathbf{M}^{-1} = a\mathbf{M} + b\mathbf{I}$$

for constants a and b to be found. *(3 marks)*

(b) (i) Determine the characteristic equation of **M** and show that **M** has a repeated eigenvalue, 1, and another eigenvalue, 2. *(6 marks)*

(ii) Give a full set of eigenvectors for each of these eigenvalues. *(5 marks)*

(iii) State the geometrical significance of each set of eigenvectors in relation to the transformation with matrix **M**. *(3 marks)*

END OF QUESTIONS

Key to mark scheme and abbreviations used in marking

M	mark is for method
m or dM	mark is dependent on one or more M marks and is for method
A	mark is dependent on M or m marks and is for accuracy
B	mark is independent of M or m marks and is for method and accuracy
E	mark is for explanation

√or ft or F	follow through from previous incorrect result	MC	mis-copy
CAO	correct answer only	MR	mis-read
CSO	correct solution only	RA	required accuracy
AWFW	anything which falls within	FW	further work
AWRT	anything which rounds to	ISW	ignore subsequent work
ACF	any correct form	FIW	from incorrect work
AG	answer given	BOD	given benefit of doubt
SC	special case	WR	work replaced by candidate
OE	or equivalent	FB	formulae book
A2,1	2 or 1 (or 0) accuracy marks	NOS	not on scheme
$-x$ EE	deduct x marks for each error	G	graph
NMS	no method shown	c	candidate
PI	possibly implied	sf	significant figure(s)
SCA	substantially correct approach	dp	decimal place(s)

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded. However, there are situations in some units where part marks would be appropriate, particularly when similar techniques are involved. Your Principal Examiner will alert you to these and details will be provided on the mark scheme.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

MFP4

Q	Solution	Marks	Total	Comments
1(a)	Rotation about the y-axis through $\cos^{-1} 0.8$	M1 A1 A1	 3	 Ignore direction or $\sin^{-1} 0.6$ or 36.87^{o} or 0.644^{c}
(b)	Reflection in $y = x$	M1A1	2	Ignore if it is called a line
	Total		**5**	
2(a)(i)	$\mathbf{a} \cdot \mathbf{b} = 0$	B1	1	
(ii)	$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 1 & 1 & -5 \end{vmatrix} = \begin{bmatrix} -16 \\ 11 \\ -1 \end{bmatrix}$	M1 A1	 2	
(iii)	$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 1 & -5 \\ 1 & 4 & 28 \end{vmatrix} = 0$	M1 A1	 2	or via $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ ft in this case Do not allow = 0 via (a)(i)
(b)(i)	$\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$ mutually perpendicular	B1	1	
(ii)	$\mathbf{a}, \mathbf{b}, \mathbf{c}$ co-planar	B1	1	
	Total		**7**	
3(a)	Area invariant $\Rightarrow$ Determinant = 1 $\Rightarrow pr + q^2 = 1$	M1 A1	 2	MUST mention area Given answer justified
(b)(i)	$\begin{bmatrix} 4 & q \\ -q & r \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\Rightarrow 2q - 4 = 2$ and $q + 2r = -1$ $\Rightarrow q = 3$ and $r = -2$	M1 A1 A1	 3	 Either correct
(ii)	$x' = 4x + 3y$ and $y' = -3x - 2y$ Setting $x' = x$, $y' = y$ $y = -x$ **Alternative for (b)(ii):** Setting $\lambda = 1$ $\Rightarrow 3x + 3y = 0$ (etc) ie $y = -x$	B1 M1 A1 (M2) (A1)	 3 (3)	
	Total		**8**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
4(a)	$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$, $\mathbf{U} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$,	B1B1		
	$\mathbf{U}^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$	B1	3	ft $\mathbf{U}^{-1}$
(b)	$\mathbf{T}^n = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$	B1 M1		For $\mathbf{D}^n$ with n even For use of $\mathbf{U}^{-1}\,\mathbf{D}^n\,\mathbf{U}$ form
	$= \begin{bmatrix} 2^n & 2\times 2^n \\ 2^n & 3\times 2^n \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$	m1 A1		
	or $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3\times 2^n & -2\times 2^n \\ -2^n & 2^n \end{bmatrix}$			
	$= 2^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	A1	5	Shown legitimately
	Alternative for (b): $\mathbf{D}^n = \begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix}$	(B1)		For $\mathbf{D}^n$ with n even
	$\mathbf{T}^n = \mathbf{U}\,(2^n\,\mathbf{I})\,\mathbf{U}^{-1}$ $= 2^n\,(\mathbf{U}\,\mathbf{I}\,\mathbf{U}^{-1})$ $= 2^n\,\mathbf{I}$	(M1) (m2) (A1)	(5)	Allow ≡ forms such as $3.2^n - 2^{n+1}$
	Total		**8**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
5(a)	eg $3\times(1)-(2) \Rightarrow 13y+13z=-13$ $(3)-(2) \Rightarrow 15y+11z=-5$	M1 A1A1		Eliminating first variable
		M1		Solving 2×2 system
	$x=6,\ y=1\frac{1}{2},\ z=-2\frac{1}{2}$	A1	5	
	Alt I (Cramer's Rule):			
	$\Delta=\begin{vmatrix}1&3&5\\3&-4&2\\3&11&13\end{vmatrix},\ \Delta_x=\begin{vmatrix}-2&3&5\\7&-4&2\\2&11&13\end{vmatrix},$			
	$\Delta_y=\begin{vmatrix}1&-2&5\\3&7&2\\3&2&13\end{vmatrix},\ \Delta_z=\begin{vmatrix}1&3&-2\\3&-4&7\\3&11&2\end{vmatrix}$	(M1)		Attempt at any two
	= 52, 312, 78 and – 130 respectively	(A1 A1)		Δ correct; ≥ 1 other determinant correct
	$x=\frac{\Delta_x}{\Delta},\ y=\frac{\Delta_y}{\Delta},\ z=\frac{\Delta_z}{\Delta}$	(M1)		At least one attempted numerically
	$x=6,\ y=1\frac{1}{2},\ z=-2\frac{1}{2}$	(A1)	(5)	
	Alt II (Augmented matrix method):			
	$\left[\begin{array}{ccc\|c}1&3&5&-2\\3&-4&2&7\\3&11&13&2\end{array}\right]\rightarrow$	(M1)		
	$\left[\begin{array}{ccc\|c}1&3&5&-2\\0&-13&-13&13\\0&2&-2&8\end{array}\right]$	(A1)		$R_2\rightarrow R_2-3R_1$ $R_3\rightarrow R_3-3R_1$
	$\rightarrow\left[\begin{array}{ccc\|c}1&3&5&-2\\0&1&1&-1\\0&1&-1&4\end{array}\right]$	(A1)		
	$\rightarrow\left[\begin{array}{ccc\|c}1&3&5&-2\\0&1&1&-1\\0&0&-2&5\end{array}\right]$			$R_3\rightarrow R_3-R_2$
	Substituting back to get $x=6,\ y=1\frac{1}{2},\ z=-2\frac{1}{2}$	(M1 A1)	(5)	
	Alt III (Inverse matrix method):			
	$C^{-1}=\frac{1}{52}\begin{bmatrix}-74&16&26\\-33&-2&13\\45&-2&-13\end{bmatrix}$	(M1) (A1 A1)		M0 if no inverse matrix is given
	$\begin{bmatrix}x\\y\\z\end{bmatrix}=C^{-1}\begin{bmatrix}-2\\7\\2\end{bmatrix}=\begin{bmatrix}6\\1.5\\-2.5\end{bmatrix}$	(M1) (A1)	(5)	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
5(b)(i)	$\begin{vmatrix} 1 & 3 & 5 \\ 3 & -4 & 2 \\ a & 11 & 13 \end{vmatrix} = 26a - 26$	M1		Attempt at determinant; OE
	Setting equal to zero and solving for a	m1		
	$a = 1$	A1	3	
(ii)	$x + 3y + 5z = -2$ $3x - 4y + 2z = 7$ $x + 11y + 13z = b$			
	NB $y + z = -1$ (from before)	B1		
	$(3) - (1) \Rightarrow 8y + 8z = b + 2$	B1		
	$b + 2 = -8 \Rightarrow b = -10$	M1A1	4	Equating; CAO
	Alternative for (b)(ii): Substituting $x = 6$, $y = 1\frac{1}{2}$, $z = -2\frac{1}{2}$			Since, to be consistent, the 3rd plane must
	into $x + 11y + 13z = b$	(M3)		contain the line of intersection of the first
	$\Rightarrow b = -10$	(A1)	(4)	2 planes, and therefore contains this point
	Total		**12**	
6(a)(i)	$\mathbf{a} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$	B1	1	
(ii)	Equating for λ: $\dfrac{x-1}{3} = \dfrac{y-1}{2} = \dfrac{z-2}{6}$	M1 A1	2	
(iii)	$\sqrt{3^2 + 2^2 + 6^2} = 7$	B1		
	Direction cosines are $\frac{3}{7}, \frac{2}{7}$ and $\frac{6}{7}$	B1		ft on 7
	These are the cosines of the angles between the line and the x- , y- and z-axes (respectively)	B1	3	Allow just "angles" correctly described
(b)(i)	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 2 \\ 1 & 1 & 3 \end{vmatrix} = 7\mathbf{i} - 10\mathbf{j} + \mathbf{k}$	M1A1		
	$d = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ -10 \\ 1 \end{bmatrix} = 0$	M1 A1	4	ft $\mathbf{n}$
(ii)	$d = 0 \Rightarrow$ plane through / contains the origin	B1	1	
(c)	$\sin\theta / \cos\theta = \dfrac{\text{scalar product}}{\text{product of moduli}}$	M1		Must be $3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ and their $\mathbf{n}$
	Numerator = 21 – 20 + 6 = 7	B1		ft correct (unsimplified)
	Denominator = $7.\sqrt{150}$	B1		ft both correct (unsimplified)
	$\theta = 4.7^\circ$	A1	4	CAO
	Total		**15**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
7(a)(i)	$\mathbf{M}^2 = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$	M1		
	$= \begin{bmatrix} 4 & -3 & 3 \\ 3 & -2 & 3 \\ 3 & -3 & 4 \end{bmatrix}$	A1		
	$\mathbf{M}^2 + 2\mathbf{I} = \begin{bmatrix} 4 & -3 & 3 \\ 3 & -2 & 3 \\ 3 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$			
	$= \begin{bmatrix} 6 & -3 & 3 \\ 3 & 0 & 3 \\ 3 & -3 & 6 \end{bmatrix} = 3\mathbf{M}$	A1	3	ie $k = 3$
(ii)	Multiplying by $\mathbf{M}^{-1}$ to get $\mathbf{M} + 2\mathbf{M}^{-1} = 3\mathbf{I}$ so that $\mathbf{M}^{-1} = \frac{3}{2}\mathbf{I} - \frac{1}{2}\mathbf{M}$	M1 A1 A1	3	ft ie $a = -\frac{1}{2}$ and $b = \frac{3}{2}$
(b)(i)	Char. eqn. is $\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$ ie $(\lambda - 2)(\lambda - 1)^2 = 0$ giving $\lambda_1 = 1$ (twice) and $\lambda_2 = 2$	M1A1 A1A1 M1 A1	6	One A mark for each of the other coefficients Good factorisation attempt
(ii)	$\lambda = 1 \Rightarrow x - y + z = 0$ (thrice) Any two independent eigenvectors (eg) $\alpha\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$	B1 M1 A1		Attempted
	$\lambda = 2 \Rightarrow -y + z = 0$ $x - 2y + z = 0 \quad \Rightarrow x = y = z$ $x - y = 0$ $\gamma\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	M1 A1	5	
(iii)	For $\lambda = 1$, eigenvectors represent a plane of invariant points For $\lambda = 2$, eigenvectors represent an invariant line	M1 A1 B1	3	Plane
	Total		**20**	
	TOTAL		**75**	

AQA – Further pure 4 – Jan 2008 – Answers

Question 1:	Exam report
$a)\begin{pmatrix}0.8 & 0 & -0.6\\ 0 & 1 & 0\\ 0.6 & 0 & 0.8\end{pmatrix}$ We notice that $0.8^2+0.6^2=1$ Let's call θ the angle so that $Cos\theta=0.8$ and $Sin\theta=0.6$ The matrix $\begin{pmatrix}0.8 & 0 & -0.6\\ 0 & 1 & 0\\ 0.6 & 0 & 0.8\end{pmatrix}=\begin{pmatrix}Cos\theta & 0 & -Sin\theta\\ 0 & 1 & 0\\ Sin\theta & 0 & Cos\theta\end{pmatrix}$ (*From formulae book* :) This this a rotation through the angle θ about the y-axis. b) $\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$ This transformation maps $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ onto $\mathbf{j}$, $\mathbf{i}$ and $\mathbf{k}$ respectively This is a reflection in the plane $y = x$.	This was a straightforward starter to the paper as might be expected, requiring candidates only to know their way round the Booklet of Formulae and interpret the standard results given therein. Centres should note that there is no intention to require candidates on this specification to specify what is meant by positive and negative directions or anti-clockwise and clockwise senses when describing rotational 3-d transformations, so no candidates were penalised for specifying such matters. Nor were candidates penalised in this case for describing *y* = *x* as a line rather than a plane in part (b). Quite a few candidates did lose a mark, however, when they failed to distinguish between 'the plane *y* = *x*' and 'the *x*-*y* plane'.

Question 2:	Exam report
$a)\, i)\, \mathbf{a}.\mathbf{b}=\begin{pmatrix}2\\3\\1\end{pmatrix}.\begin{pmatrix}1\\1\\-5\end{pmatrix}=2+3-5=0$ $\mathbf{a}$ *and* $\mathbf{b}$ *are perpendicular* $ii)\, \mathbf{a}\times\mathbf{b}=\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 3 & 1\\ 1 & 1 & -5\end{pmatrix}=\begin{vmatrix}3 & 1\\ 1 & -5\end{vmatrix}\mathbf{i}-\begin{vmatrix}2 & 1\\ 1 & -5\end{vmatrix}\mathbf{j}+\begin{vmatrix}2 & 3\\ 1 & 1\end{vmatrix}\mathbf{k}$ $=-16\mathbf{i}+11\mathbf{j}-\mathbf{k}$ $iii)\, \mathbf{a}.(\mathbf{b}\times\mathbf{c})=\begin{pmatrix}2\\3\\1\end{pmatrix}.\begin{pmatrix}-16\\11\\-1\end{pmatrix}=-32+33-1=0$ $b)\, i)\, \mathbf{a},\mathbf{b}$ *and* $\mathbf{a}\times\mathbf{b}$ are perpendicular to each other. $ii)\, \mathbf{a}.(\mathbf{b}\times\mathbf{c})=0$ *so* $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar.	This was another fairly gentle starter to the paper, with most candidates scoring at least six of the seven marks. Where marks were lost, this was usually in failing to grasp what was required by the request for 'geometrical' interpretations in part (b) – as was also the case in questions 3, 6 and 7. Here, candidates were expected to point out that **a**, **b and a × b** are perpendicular to each other. Many simply noted that the vector product was perpendicular to both **a** and **b**. Those candidates who simply stated 'perpendicular' as a one-word answer were not given the benefit of the doubt, as there were many others who essentially said just this, but failed to be complete when going on to describe what was at right-angles to what. Many others missed the mark in part (b)(ii) by describing the algebraic relationship of linear dependence but failing to note the geometrical relationship of co-planarity.

Question 3:	Exam report
$\mathbf{A}=\begin{bmatrix} p & q \\ -q & r \end{bmatrix}$ $a)$ Through a shear, a shape is transformed into a shape with the same area, this mean that det(**A**)=1 $\det(\mathbf{A})=\begin{vmatrix} p & q \\ -q & r \end{vmatrix}=pr+q^2=1$ $b)\ p=4 \text{ so } \mathbf{A}=\begin{pmatrix} 4 & q \\ -q & r \end{pmatrix}$ $and \begin{pmatrix} 4 & q \\ -q & r \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}=\begin{pmatrix} -4+2q \\ q+2r \end{pmatrix}=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ This gives $q=3$ *and* $r=-2$ $ii)$ Let's call the equation of this invariant line $y=mx$ (The line goes through O(0,0) because O is invariant) $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix}=\begin{pmatrix} 4x+3mx \\ -3x-2mx \end{pmatrix}=\begin{pmatrix} x \\ mx \end{pmatrix}$ This gives $4+3m=1$ *and* $-3-2m=m$ The equation are consistent and $m=-1$ The invariant line has equation $y=-x$	As in Question 2, many answers failed to point out in part (a) that the geometrical property of a shear required to explain the given result is the invariance of areas under the transformation. Otherwise, the question was very capably handled.

Question 4:	Exam report
$a)\ \mathbf{U}=\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \text{ and } \mathbf{D}=\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ $\det(\mathbf{U})=3-2=1 \text{ so } \mathbf{U}^{-1}=\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$ $b)\ \mathbf{T}=\mathbf{UDU}^{-1} \text{ so } \mathbf{T}^n=\mathbf{UD}^n\mathbf{U}^{-1}$ $\mathbf{T}^n=\mathbf{U}\begin{pmatrix} 2^n & 0 \\ 0 & (-2)^n \end{pmatrix}\mathbf{U}^{-1} \text{ and when } n \text{ is EVEN}$ $\mathbf{T}^n=\mathbf{U}\begin{pmatrix} 2^n & 0 \\ 0 & 2^n \end{pmatrix}\mathbf{U}^{-1}=\mathbf{U}\times 2^n\mathbf{I}\times\mathbf{U}^{-1}=2^n\mathbf{UIU}^{-1}=2^n\mathbf{I}$ $f(n)=2^n$	This question was actually quite straightforward. However, apart from the final parts of Question 7, it managed to present the biggest obstacle on the paper. Surprisingly, the greatest difficulty arose when candidates were required to turn $(-2)^n$ into 2^n, in the case when *n* is even, when considering the matrix $\mathbf{D}^n$. Many failed to manage this at all. A disappointingly large number of candidates left it until well into their working and many such candidates then failed to cope with the complicated expressions they were trying to deal with. Those that did so at the outset found themselves with an alternative and simple route through the problem available to them, since it follows that $\mathbf{D}^n = 2^n\ \mathbf{I}$; it is now straightforward to re-write $\mathbf{T}^n$ and to note that f(*n*) = 2^n.

Question 5:	Exam report
When $a=3$ *and* $b=2$, the system of equation becomes $\begin{cases} x+3y+5z=-2 & l_1 \\ 3x-4y+2z=7 & l_2 \\ 3x+11y+13z=2 & l_3 \end{cases} \Leftrightarrow \begin{cases} x+3y+5z=-2 \\ 13y+13z=-13 \ (3l_1-l_2) \\ -2y+2z=-8 \quad (3l_1-l_3) \end{cases}$ $\Leftrightarrow \begin{cases} x+3y+5z=-2 \\ y+z=-1 \\ -y+z=-4 \end{cases} \Leftrightarrow \begin{cases} x+3y+5z=-2 \\ y+z=-1 \\ 2z=-5 \end{cases} \Leftrightarrow \begin{cases} x=-2-3y-5z=6 \\ y=-1-z=\frac{3}{2} \\ z=-\frac{5}{2} \end{cases}$ The unique solution is $(6, \frac{3}{2}, -\frac{5}{2})$ $b\,i)$ The system does not have a unique solution when the determinant of the associated matrix $=0$ $\det = \begin{vmatrix} 1 & 3 & 5 \\ 3 & -4 & 2 \\ a & 11 & 13 \end{vmatrix} = \begin{vmatrix} -4 & 2 \\ 11 & 13 \end{vmatrix} - 3\begin{vmatrix} 3 & 2 \\ a & 13 \end{vmatrix} + 5\begin{vmatrix} 3 & -4 \\ a & 11 \end{vmatrix}$ $\det = -74-3(39-2a)+5(33+4a)=26a-26$ $\det = 0$ for $a=1$. $ii)$ When $a=1$, the system becomes $\begin{cases} x+3y+5z=-2 & l_1 \\ 3x-4y+2z=7 & l_2 \\ x+11y+13z=b & l_3 \end{cases} \Leftrightarrow \begin{cases} x+3y+5z=-2 \\ 13y+13z=-13 \ (3l_1-l_2) \\ -8y-8z=-2-b \ (l_1-l_2) \end{cases}$ $\Leftrightarrow \begin{cases} x+3y+5z=-2 \\ y+z=-1 \\ y+z=\frac{b+2}{8} \end{cases}$ The system is consistent when $\frac{b+2}{8}=-1$ The system is consistent when $b=-10$.	It was very pleasing to see a large numbers of candidates solving part (a)'s system of equations using some fairly high-powered techniques (see the end of the marking scheme for these: Cramer's Rule, the augmented matrix method, and the inverse matrix method) to find the unique solution, but most of these are actually more complicated than what was required. The "low-tech" algebraic approach does the job so easily that the others (whilst eminently worth teaching) often turn out to be longer and more prone to errors. One approach that did turn up for part (b)(ii) is that one can simply substitute the values of *x*, *y* and *z* found in part (a) into the 3rd equation – using the value of *a* from part (b)(i) – in order to find the value of *b* required for the final equation to be consistent. This is because the third plane must share the line of intersection of the first two planes, and hence the point on it found in part (a). Only a handful of students actually employed this approach.

Question 6:	Exam report
$a)\ i)$ A direction vector of the line is $\mathbf{u}=\begin{pmatrix}3\\2\\6\end{pmatrix}$ The point A with position $\mathbf{a}=\begin{pmatrix}1\\1\\2\end{pmatrix}$ belongs to the line . A vector equation is $\left(\mathbf{r}-\begin{pmatrix}1\\1\\2\end{pmatrix}\right)\times\begin{pmatrix}3\\2\\6\end{pmatrix}=0$ $ii)$ Cartesians equations of this line : $\frac{x-1}{3}=\frac{y-1}{2}=\frac{z-2}{6}$ $iii)\ \lvert\mathbf{u}\rvert=\sqrt{3^2+2^2+6^2}=\sqrt{49}=7$ The direction cosines are: $\frac{3}{7},\frac{2}{7},\frac{6}{7}$ Theses are the cosine of the angles the line makes with the x, y and z -axes. b) i) A normal vector to the plane is $\mathbf{n}=\begin{pmatrix}4\\3\\2\end{pmatrix}\times\begin{pmatrix}1\\1\\3\end{pmatrix}$ $\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&3&2\\1&1&3\end{vmatrix}=7\mathbf{i}-10\mathbf{j}+\mathbf{k}$ An equation of the line is $\left(\mathbf{r}-\begin{pmatrix}7\\5\\1\end{pmatrix}\right).\begin{pmatrix}7\\-10\\1\end{pmatrix}=0$ $\mathbf{r}.\begin{pmatrix}7\\-10\\1\end{pmatrix}=\begin{pmatrix}7\\5\\1\end{pmatrix}.\begin{pmatrix}7\\-10\\1\end{pmatrix}=0\ ,\ \mathbf{r}.\begin{pmatrix}7\\-10\\1\end{pmatrix}=0$ $ii)$ The plane is going through the origin. c) Let's work out the angle θ bewteen $\mathbf{u}=\begin{pmatrix}3\\2\\6\end{pmatrix}$ and $\mathbf{n}=\begin{pmatrix}7\\-10\\1\end{pmatrix}$ $\mathbf{u.n}=\begin{pmatrix}3\\2\\6\end{pmatrix}.\begin{pmatrix}7\\-10\\1\end{pmatrix}=21-20+6=7\ ,\ \lvert\mathbf{u}\rvert=7\ and\ \lvert\mathbf{n}\rvert=\sqrt{150}$ $Cos\theta=\frac{7}{7\sqrt{150}}=\frac{1}{\sqrt{150}}\quad so\ \theta=85.3^o$ The angle between the line and the plane is $90^o-85.3^o=4.7^0$	This question was handled very confidently by most candidates, and most were aware of the geometrical significance of the direction cosines of a line. Far fewer appreciated that the value d = 0 meant that part (b)'s plane contained the origin; many had a stab at some perpendicularity arrangement instead. In part (c) a significant minority of candidates found the complementary angle to the one required.

Question 7:

$$\mathbf{M}=\begin{pmatrix}2&-1&1\\1&0&1\\1&-1&2\end{pmatrix} \text{ and } \mathbf{M}^2=\begin{pmatrix}2&-1&1\\1&0&1\\1&-1&2\end{pmatrix}\times\begin{pmatrix}2&-1&1\\1&0&1\\1&-1&2\end{pmatrix}=\begin{pmatrix}4&-3&3\\3&-2&3\\3&-3&4\end{pmatrix}$$

$$\mathbf{M}^2+2\mathbf{I}=\begin{pmatrix}4&-3&3\\3&-2&3\\3&-3&4\end{pmatrix}+\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}=\begin{pmatrix}6&-3&3\\3&0&3\\3&-3&6\end{pmatrix}=3\mathbf{M}$$

$k=3$

$ii)\ \mathbf{M}^2+2\mathbf{I}=3\mathbf{M}\quad(\times\mathbf{M}^{-1})$

$\mathbf{M}+2\mathbf{M}^{-1}=3\mathbf{I}$

$\mathbf{M}^{-1}=\frac{3}{2}\mathbf{I}-\frac{1}{2}\mathbf{M}\qquad a=-\frac{1}{2} \text{ and } b=\frac{3}{2}$

$$b)\ \det(\mathbf{M}-\lambda\mathbf{I})=\begin{vmatrix}2-\lambda&-1&1\\1&-\lambda&1\\1&-1&2-\lambda\end{vmatrix}=\begin{vmatrix}2-\lambda&0&1\\1&1-\lambda&1\\1&1-\lambda&2-\lambda\end{vmatrix}$$

$$\det(\mathbf{M}-\lambda\mathbf{I})=(1-\lambda)\begin{vmatrix}2-\lambda&0&1\\1&1&1\\1&1&2-\lambda\end{vmatrix}=(1-\lambda)\begin{vmatrix}2-\lambda&0&1\\0&0&\lambda-1\\1&1&2-\lambda\end{vmatrix}$$

$\det(\mathbf{M}-\lambda\mathbf{I})=(1-\lambda)(2-\lambda)(\lambda-1)$

$\det(\mathbf{M}-\lambda\mathbf{I})=0\ \ for\ \ \lambda=1(repeated\ value)\ and\ \lambda=2$

$$ii)\ For\ \lambda=1, let's\ solve\ (\mathbf{M}-\mathbf{I})\begin{pmatrix}x\\y\\z\end{pmatrix}=0\Leftrightarrow\begin{pmatrix}1&-1&1\\1&-1&1\\1&-1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=0$$

$$\Leftrightarrow\begin{cases}x-y+z=0\\x-y+z=0\\x-y+z=0\end{cases}\quad \text{An eigenvector is }\begin{pmatrix}0\\1\\1\end{pmatrix} \text{ and another one is }\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

(Make sure that the vector are linearly independent)

$$For\ \lambda=2, let's\ solve\ (\mathbf{M}-2\mathbf{I})\begin{pmatrix}x\\y\\z\end{pmatrix}=0\Leftrightarrow\begin{pmatrix}0&-1&1\\1&-2&1\\1&-1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=0$$

$$\Leftrightarrow\begin{cases}-y+z=0\\x-2y+z=0\\x-y=0\end{cases}\Leftrightarrow\begin{cases}x=y\\y=z\end{cases}\quad \text{An eigenvector is }\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

$iii)\ For\ \lambda=1,$

the eigenvectors are direction vectors of a plane of invariant points.

For $\lambda=2$,

the eigenvector is a direction vector of an invariant line.

Exam report

This question was definitely the most demanding one on the paper; partly for its length, and partly due to working with a 3 × 3 matrix. On top of this, there was a repeated eigenvalue to the matrix, and a lot of candidates clearly did not know quite what to make of it. Nonetheless, even the relatively weaker candidates were still able to gain up to 12 of the marks.

Part (a)(i) was very well handled by those who attempted it. Part (a)(ii) was less well handled. Many of those who fell down here did so because they then went off to find the inverse matrix $\mathbf{M}^{-1}$ itself, rather than follow the logic of the question. Many more who did multiply the matrix equation by $\mathbf{M}^{-1}$ got tangled up in some way – failing to spot that $\mathbf{M}\ \mathbf{M}^{-1}=\mathbf{I}$, for instance, or being unable to divide by 2 under exam conditions.

In part (b), the factorisation of the determinant $|\ \mathbf{M}-\lambda\ \mathbf{I}\ |$ was usually approached by row/column operations, which was very pleasing to see. However, it frequently (actually usually) led to the strange sight of candidates taking a partially or completely factorised expression, and then multiplying it out to get a standard cubic form, only to start to factorise it again to find the three eigenvalues.

Those who made a mistake in this working then shot themselves in the foot, losing the following method mark for the attempt to factorise, which they failed to pick up by presuming that their cubic actually *was* equal to the factorised form $(\lambda-1)^2(\lambda-2)$ rather than doing any of the necessary working or going back to check their previous, incorrect working.

The final two parts to the question were demanding, but actually not as difficult as candidates found them to be. Despite the fact that most candidates getting this far quite correctly deduced that $x-y+z=0$ for the repeated eigenvalue $\lambda=1$, almost all of them then believed this represented the equation of a line. This error then prevented them from finding a pair of independent representative eigenvectors (or an alternative parametric plane form) and also from describing the result as a plane of invariant points, rather than as a line.

Grade boundaries

Grade			A	B	C	D	E
Mark	Max 75		60	52	44	36	29