General Certificate of Education
January 2007
Advanced Level Examination

MATHEMATICS **MFP4**
Unit Further Pure 4

Wednesday 31 January 2007 9.00 am to 10.30 am

For this paper you must have:
- an 8-page answer book
- the **blue** AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP4.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 Show that the system of equations

$$\begin{aligned} x + 2y - z &= 0 \\ 3x - y + 4z &= 7 \\ 8x + y + 7z &= 30 \end{aligned}$$

is inconsistent. *(4 marks)*

2 (a) Show that $(a - b)$ is a factor of the determinant

$$\Delta = \begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ bc & ca & ab \end{vmatrix}$$

(2 marks)

(b) Factorise Δ completely into linear factors. *(5 marks)*

3 The points P, Q and R have position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively relative to an origin O, where

$$\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \ \mathbf{q} = \begin{bmatrix} -3 \\ 4 \\ 20 \end{bmatrix} \text{ and } \mathbf{r} = \begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix}$$

(a) (i) Determine $\mathbf{p} \times \mathbf{q}$. *(2 marks)*

(ii) Find the area of triangle OPQ. *(3 marks)*

(b) Use the scalar triple product to show that $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are linearly dependent, and interpret this result geometrically. *(3 marks)*

4 The matrices $\mathbf{M}_{\mathrm{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\mathbf{M}_{\mathrm{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ represent the transformations A and B respectively.

(a) Give a full geometrical description of each of A and B. *(5 marks)*

(b) Transformation C is obtained by carrying out A followed by B.

(i) Find $\mathbf{M}_{\mathrm{C}}$, the matrix of C. *(2 marks)*

(ii) Hence give a full geometrical description of the single transformation C. *(2 marks)*

5 (a) Find, to the nearest 0.1°, the acute angle between the planes with equations

$$\mathbf{r} \cdot (3\mathbf{i} - 4\mathbf{j} + \mathbf{k}) = 2 \quad \text{and} \quad \mathbf{r} \cdot (2\mathbf{i} + 12\mathbf{j} - \mathbf{k}) = 38$$ *(4 marks)*

(b) Write down cartesian equations for these two planes. *(2 marks)*

(c) (i) Find, in the form $\dfrac{x-a}{l} = \dfrac{y-b}{m} = \dfrac{z-c}{n}$, cartesian equations for the line of intersection of the two planes. *(5 marks)*

(ii) Determine the direction cosines of this line. *(2 marks)*

6 (a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$$ *(6 marks)*

(b) (i) Write down a diagonal matrix **D**, and a suitable matrix **U**, such that

$$\mathbf{X} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$$ *(2 marks)*

(ii) Write down also the matrix $\mathbf{U}^{-1}$. *(1 mark)*

(iii) Use your results from parts (b)(i) and (b)(ii) to determine the matrix $\mathbf{X}^5$ in the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c and d are integers. *(3 marks)*

Turn over for the next question

7 The transformation S is a shear with matrix $\mathbf{M} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$. Points (x, y) are mapped under S to image points (x', y') such that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

(a) Find the equation of the line of invariant points of S. *(2 marks)*

(b) Show that all lines of the form $y = x + c$, where c is a constant, are invariant lines of S. *(3 marks)*

(c) Evaluate $\det \mathbf{M}$, and state the property of shears which is indicated by this result. *(2 marks)*

(d) Calculate, to the nearest degree, the acute angle between the line $y = -x$ and its image under S. *(3 marks)*

8 The matrix $\mathbf{P} = \begin{bmatrix} 4 & -1 & 2 \\ 1 & 1 & 3 \\ -2 & 0 & a \end{bmatrix}$, where a is constant.

(a) (i) Determine $\det \mathbf{P}$ as a linear expression in a. *(2 marks)*

(ii) Evaluate $\det \mathbf{P}$ in the case when $a = 3$. *(1 mark)*

(iii) Find the value of a for which $\mathbf{P}$ is singular. *(2 marks)*

(b) The 3×3 matrix $\mathbf{Q}$ is such that $\mathbf{PQ} = 25\mathbf{I}$.

Without finding $\mathbf{Q}$:

(i) write down an expression for $\mathbf{P}^{-1}$ in terms of $\mathbf{Q}$; *(1 mark)*

(ii) find the value of the constant k such that $(\mathbf{PQ})^{-1} = k\mathbf{I}$; *(2 marks)*

(iii) determine the numerical value of $\det \mathbf{Q}$ in the case when $a = 3$. *(4 marks)*

END OF QUESTIONS

Key to mark scheme and abbreviations used in marking

M	mark is for method
m or dM	mark is dependent on one or more M marks and is for method
A	mark is dependent on M or m marks and is for accuracy
B	mark is independent of M or m marks and is for method and accuracy
E	mark is for explanation

√or ft or F	follow through from previous incorrect result	MC	mis-copy
CAO	correct answer only	MR	mis-read
CSO	correct solution only	RA	required accuracy
AWFW	anything which falls within	FW	further work
AWRT	anything which rounds to	ISW	ignore subsequent work
ACF	any correct form	FIW	from incorrect work
AG	answer given	BOD	given benefit of doubt
SC	special case	WR	work replaced by candidate
OE	or equivalent	FB	formulae book
A2,1	2 or 1 (or 0) accuracy marks	NOS	not on scheme
$-x$ EE	deduct x marks for each error	G	graph
NMS	no method shown	c	candidate
PI	possibly implied	sf	significant figure(s)
SCA	substantially correct approach	dp	decimal place(s)

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded. However, there are situations in some units where part marks would be appropriate, particularly when similar techniques are involved. Your Principal Examiner will alert you to these and details will be provided on the mark scheme.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

MFP4

Q	Solution	Marks	Total	Comments
1	$\left(\begin{array}{ccc\|c} 1 & 2 & -1 & 0 \\ 3 & -1 & 4 & 7 \\ 8 & 1 & 7 & 30 \end{array}\right) \to \left(\begin{array}{ccc\|c} 1 & 2 & -1 & 0 \\ 0 & -1 & 7 & 7 \\ 0 & -15 & 15 & 30 \end{array}\right)$	M1 A1		$R_2' = R_2 - 3R_1$ $R_3' = R_3 - 8R_1$ Penalise numerical errors once only, at this stage
	$\to \left(\begin{array}{ccc\|c} 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 2 \end{array}\right)$	A1 B1F	4	Inconsistency noted/explained ft provided working is clear
	Or $\Delta = -7 - 3 + 64 - 8 - 4 - 42 = 0$ and Δ_x or Δ_y or $\Delta_z = 0$ shown also Explaining this $\Rightarrow$ inconsistency	(M1) (A1) (A1) (B1)	(4)	So showing $\Delta = 0$ and thinking this is it scores M1A1A0B0
	Or Solving **(1)** & **(2)**, say, to get $x = \lambda,\ y = 1 - \lambda,\ z = 2 - \lambda$ Subst^{g}. in **(3)** $\Rightarrow$ $15 = 30$	(M1) (A1) (A1) (B1)	(4)	Checking to show inconsistency
	Total		**4**	
2(a)	$\Delta = \begin{vmatrix} a-b & b & c \\ b-a & c+a & a+b \\ c(b-a) & ca & ab \end{vmatrix}$	M1		$C_1' = C_1 - C_2$
	$= (a-b)\begin{vmatrix} 1 & b & c \\ -1 & c+a & a+b \\ -c & ca & ab \end{vmatrix}$	A1	2	
	Or Setting $b = a \Rightarrow C_1 = C_2 \Rightarrow \Delta = 0$ $\Rightarrow (a-b)$ a factor of Δ	(M1) (A1)	(2)	Factor theorem
	Or $\Delta = (a-b)(c^3 + a^2b + ab^2 - abc - b^2c - a^2c)$	(M1) (A1)	(2)	Must be completely correct

MFP4 (cont)

Q	Solution	Marks	Total	Comments
2(b)	$= (a-b)\begin{vmatrix} 1 & b-c & c \\ -1 & c-b & a+b \\ -c & a(c-b) & ab \end{vmatrix}$	M1		$C_2' = C_2 - C_3$
	$= (a-b)(b-c)\begin{vmatrix} 1 & 1 & c \\ -1 & -1 & a+b \\ -c & -a & ab \end{vmatrix}$	A1		2nd linear factor extracted
	e.g. $\Delta = (a-b)(b-c)\begin{vmatrix} 0 & 0 & a+b+c \\ -1 & -1 & a+b \\ -c & -a & ab \end{vmatrix}$	M1		Genuine attempt at both remaining linear factors: e.g. $R_1' = R_1 + R_2$
	and then expanding final det.	A1		3rd factor
	$\Delta = -(a+b+c)(a-b)(b-c)(c-a)$	A1	5	All correct
	Or By cyclic symmetry, $(b-c)$ and $(c-a)$ are also factors	(M1) (A1) (A1)		
	Final linear factor & checking sign of a coefficient.	(M1) (A1)	(5)	
	Or Expanding the determinant fully	(M1)		
	$\Delta =$	(A1)		
	Multiplying out $(a-b)(b-c)(c-a)(a+b+c)$	(M1)		No fudging, or jumping straight to the answer allowed
	$=$	(A1)		
	Fully correct working to show the two things are identically equal & checking for sign	(A1)	(5)	
	Total		**7**	
3(a)(i)	$\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 4 \\ -3 & 4 & 20 \end{vmatrix} = \begin{bmatrix} 4 \\ -32 \\ 7 \end{bmatrix}$	M1 A1	2	
(ii)	$A = \frac{1}{2} \lvert \mathbf{p} \times \mathbf{q} \rvert$	M1		
	$= \frac{1}{2}\sqrt{4^2 + 32^2 + 7^2}$	B1		For attempt at $\lvert \mathbf{p} \times \mathbf{q} \rvert$
	$= \frac{33}{2}$	A1F	3	ft
(b)	$\mathbf{p} \times \mathbf{q} \,.\, \mathbf{r} = \begin{bmatrix} 4 \\ -32 \\ 7 \end{bmatrix} . \begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix}$ *or* $\begin{vmatrix} 1 & 1 & 4 \\ -3 & 4 & 20 \\ 9 & 2 & 4 \end{vmatrix}$	M1		
	$= 36 - 64 + 28 = 0$	A1		Give when "= 0" reached
	($\Rightarrow$ Lin Dep) *O, P, Q, R* **Or p, q, r** co-planar	B1	3	
	Total		**8**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
4(a)	A is a Rotation thro' 90°	M1 A1		
	about Ox	A1		
	B is a Reflection in $y = 0$ (i.e. x–z plane)	M1 A1	5	
(b)(i)	$\mathbf{M}_C = \mathbf{M}_B\,\mathbf{M}_A = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}$	M1 A1	2	
(ii)	C is a Reflection in $y = z$	M1	2	Give M1 for any series of reflections
		A1		
	N.B. In (i):			
	$\mathbf{M}_A\,\mathbf{M}_B = \begin{bmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & -1 & 0\end{bmatrix}$ scores M0			
	but ft "Reflection in $y = -z$" in (ii)			
			9	
5(a)	Use of $\cos\theta = \dfrac{\text{scalar product}}{\text{product of moduli}}$	M1		Must be $(3\mathbf{i} - 4\mathbf{j} + \mathbf{k})$ and $(2\mathbf{i} + 12\mathbf{j} - \mathbf{k})$
	Numerator = ± 43	B1		
	Denominator = $\sqrt{26}\,.\,\sqrt{149}$	B1		Dr. = 5.099... × 3.742... = 0.6908...
	$\theta = 46.3°$	A1	4	
(b)	$3x - 4y + z = 2$ and $2x + 12y - z = 38$	B1 B1	2	
(c)(i)	$(3\mathbf{i} - 4\mathbf{j} + \mathbf{k}) \times (2\mathbf{i} + 12\mathbf{j} - \mathbf{k})$	M1		
	$= -8\mathbf{i} + 5\mathbf{j} + 44\mathbf{k}$	A1		
	p.v. of any point on line	M1		
	e.g. (0, 5, 22), (8, 0, – 22), $(4, 2\frac{1}{2}, 0)$	A1		
	$\dfrac{x - x_c}{-8} = \dfrac{y - y_c}{5} = \dfrac{z - z_c}{44}$	B1F	5	ft
	Or Adding ⇒ $5x + 8y = 40$ (e.g.)	(M1)		Eliminating one variable
	$\dfrac{x-8}{-8} = \dfrac{y}{5} = \lambda$ Or $\dfrac{x}{-8} = \dfrac{y-5}{5} = \mu$	(dM1)		Parametrisation attempted
		(A1)		
	$x = 8 - 8\lambda$, $x = -8\mu$			
	$y = 5\lambda$, $y = 5 + 5\mu$	(M1)		Subst[g]. to find third variable
	⇒ $z = 44\lambda - 22$ ⇒ $z = 44\lambda + 22$			
	$\dfrac{x - x_c}{-8} = \dfrac{y - y_c}{5} = \dfrac{z - z_c}{44}$	(A1)	(5)	
(ii)	$\sqrt{8^2 + 5^2 + 44^2} = 45$	B1F		ft
	d.c.s are $\dfrac{-8}{45}$, $\dfrac{1}{9}$ and $\dfrac{44}{45}$	B1F	2	ft
	Total		**13**	

MFP4 (cont)

Q	Solution	Marks	Total	Comments
6(a)	Char. Eqn. is $\lambda^2 - 5\lambda - 6 = 0$	B1		
	Solving $\Rightarrow \lambda = -1$ or 6	M1 A1		
	Subst^g. either λ back	M1		
	$\lambda = -1 \Rightarrow x + y = 0 \Rightarrow$ evecs. $\alpha\begin{bmatrix}1\\-1\end{bmatrix}$	A1		Any non-zero α
	$\lambda = 6 \Rightarrow 5x - 2y = 0 \Rightarrow$ evecs. $\beta\begin{bmatrix}2\\5\end{bmatrix}$	A1	6	Any non-zero β
(b)(i)	$\mathbf{D} = \begin{bmatrix}-1 & 0\\0 & 6\end{bmatrix}$ $\quad \mathbf{U} = \begin{bmatrix}1 & 2\\-1 & 5\end{bmatrix}$	B1F B1F	2	ft evals. ft evecs. (must correspond to their evals.)
(ii)	$\mathbf{U}^{-1} = \frac{1}{7}\begin{bmatrix}5 & -2\\1 & 1\end{bmatrix}$	B1F	1	ft their **U** (provided non-singular)
(iii)	$\mathbf{X}^5 = \mathbf{U}\,\mathbf{D}^5\,\mathbf{U}^{-1}$	M1		
	$= \frac{1}{7}\begin{bmatrix}1 & 2\\-1 & 5\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & 6^5\end{bmatrix}\begin{bmatrix}5 & -2\\1 & 1\end{bmatrix}$	B1F		Use of correct $\mathbf{D}^5$ (ft) N.B. $6^5 = 7776$
	$= \begin{bmatrix}2221 & 2222\\5555 & 5554\end{bmatrix}$	A1	3	
			12	
7(a)	Setting $x' = x$ and $y' = y$ $x = -x + 2y$ and $y = -2x + 3y$	M1		Or via evals/evecs
	gives $y = x$	A1	2	
(b)	$\begin{bmatrix}-1 & 2\\-2 & 3\end{bmatrix}\begin{bmatrix}x\\x+c\end{bmatrix} = \begin{bmatrix}x+2c\\x+3c\end{bmatrix}$	M1A1		
	And $y' = x' + c$ also	B1	3	Explanation
(c)	det $\mathbf{M} = 1 \Rightarrow$ Areas of shapes invariant	B1 B1	2	
(d)	$\begin{bmatrix}-1 & 2\\-2 & 3\end{bmatrix}\begin{bmatrix}a\\-a\end{bmatrix} = \begin{bmatrix}-3a\\-5a\end{bmatrix}$	M1		
	$\Rightarrow$ Image of $y = -x$ under S is $y = \frac{5}{3}x$	A1		
	Angle is $135° - \tan^{-1}\frac{5}{3} = 76°$	B1F	3	ft
	N.B. Final angle can be gained via scalar product: $\cos\theta = \left\lvert\frac{(\mathbf{i}-\mathbf{j})\cdot(-3\mathbf{i}-5\mathbf{j})}{\sqrt{2}\sqrt{34}}\right\rvert$ $\Rightarrow \theta = \cos^{-1}(1/\sqrt{17}) = 76°$			
	Total		**10**	

MFP4 (Cont)

Q	Solution	Marks	Total	Comments
8(a)(i)	det $\mathbf{P} = 4a + 6 + 4 + a = 5a + 10$	M1 A1	2	
(ii)	When $a = 3$, det $\mathbf{P} = 25$	B1F	1	ft
(iii)	Setting their det $\mathbf{P} = 0 \Rightarrow a = -2$	M1 A1F	 2	 ft
(b)(i)	$\mathbf{P}^{-1} = \frac{1}{25}\mathbf{Q}$	B1	1	
(ii)	$(\mathbf{PQ})^{-1} = (25\,\mathbf{I})^{-1} = \frac{1}{25}\mathbf{I}$	M1 A1	2	
	Or $(\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}$	(M1)		Ignore $(\mathbf{PQ})^{-1} = \mathbf{P}^{-1}\mathbf{Q}^{-1}$ if they can make it work
	$= \mathbf{Q}^{-1} . \frac{1}{25}\mathbf{Q} = \frac{1}{25}\mathbf{I}$	(A1)	(2)	
(iii)	det $\mathbf{PQ}$ = det $(25\,\mathbf{I}) = 25^3$ or 15625 det $\mathbf{PQ}$ = det $\mathbf{P}$. det $\mathbf{Q}$ $\Rightarrow 25^3 = 25$ det $\mathbf{Q}$ $\Rightarrow$ det $\mathbf{Q} = 25^2$ or 625	M1 A1 M1 A1	 4	 Used
	Total		**12**	
	TOTAL		**75**	

AQA – Further pure 4 – Jan 2007 – Answers

Question 1:	Exam report
$\begin{cases} x+2y-z=0 & l_1 \\ 3x-y+4z=7 & l_2 \\ 8x+y+7z=30 & l_3 \end{cases} \Leftrightarrow \begin{cases} 3x-y+4z=7 & l_2 \\ 11x+11z=37 & l_2+l_3 \\ 7x+7z=14 & 2l_2+l_1 \end{cases}$ $\Leftrightarrow \begin{cases} 3x-y+4z=7 \\ x+z=\frac{37}{11} \\ x+z=2 \end{cases}$ The system is inconsistent.	Although intended as a straightforward starter, this question caused as much difficulty as any on the paper. Around a third of candidates thought that it was sufficient to show that the determinant of the coefficient matrix was zero – ie for non-uniqueness of solution(s) – while many others relied on faulty arithmetic to support their claim of inconsistency.

Question 2:	Exam report
$\Delta = \begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} a-b & b & c \\ b-a & c+a & a+b \\ c(b-a) & ca & ab \end{vmatrix} \quad (C_1' = C_1 - C_2)$ $\Delta = (a-b)\begin{vmatrix} 1 & b & c \\ -1 & c+a & a+b \\ -c & ca & ab \end{vmatrix} = (a-b)\begin{vmatrix} 1 & b & c \\ 0 & a+b+c & a+b+c \\ -c & ca & ab \end{vmatrix}$ $\Delta = (a-b)(a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & 1 & 1 \\ -c & ca & ab \end{vmatrix} = (a-b)(a+b+c)\begin{vmatrix} 1 & b & c-b \\ 0 & \boxed{1} & 0 \\ -c & ca & ab-ca \end{vmatrix}$ $\Delta = (a-b)(a+b+c)(ab-ca+c(c-b)) = (a-b)(a+b+c)(a(b-c)-c(b-c))$ $\Delta = (a-b)(a+b+c)(b-c)(a-c)$	This was the type of question previously handled very badly, but this time most candidates appeared well-drilled in how to manipulate determinants. The greatest obstacles to a completely successful conclusion to the question lay in the widespread instinct to expand the remaining determinant at too early a stage, leaving many candidates unable to factorise a quadratic expression in two variables, for instance; and in the more minor error of assuming that the final expression was fully cyclically symmetric in *a*, *b*, *c*, but failing to check that there was a 'minus sign' difference between what they were expecting and what they had been given.

Question 3:	Exam report
$a)\, i)\, \mathbf{p}\times\mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 4 \\ -3 & 4 & 20 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 4 & 20 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 4 \\ -3 & 20 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ -3 & 4 \end{vmatrix}\mathbf{k}$ $\mathbf{p}\times\mathbf{q} = 4\mathbf{i} - 32\mathbf{j} + 7\mathbf{k}$ $ii)$ Area of OPQ$=\frac{1}{2}\lvert\mathbf{p}\times\mathbf{q}\rvert = \frac{1}{2}\sqrt{4^2+(-32)^2+7^2} = \frac{33}{2}$ $b)\, \mathbf{p}.(\mathbf{q}\times\mathbf{r}) = \begin{vmatrix} 1 & 1 & 4 \\ -3 & 4 & 20 \\ 9 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 4 & 20 \\ 2 & 4 \end{vmatrix} - \begin{vmatrix} -3 & 20 \\ 9 & 4 \end{vmatrix} + 4\begin{vmatrix} -3 & 4 \\ 9 & 2 \end{vmatrix}$ $\mathbf{p}.(\mathbf{q}\times\mathbf{r}) = 16-40+12+180-24-144 = 0$ The vector $\mathbf{p},\mathbf{q}$ and $\mathbf{r}$ are coplanar or The points O, P Q and R are coplanar	This question was handled very well indeed, apart from minor slips in the arithmetic. The geometrical interpretation of linear dependence sought was that **p**, **q** and **r** were co-planar. Many candidates eased their uncertainty as to what was required by saying it in several different ways.

Question 4:	Exam report
$\mathbf{M}_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & Cos(90) & -Sin(90) \\ 0 & Sin(90) & Cos(90) \end{pmatrix}$. This is a rotation 90^0 *about the x − axis*. $\mathbf{M}_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ This transformation maps the base vectors as such $\mathbf{i} \rightarrow \mathbf{i}$ $\mathbf{j} \rightarrow -\mathbf{j}$ This is a reflection in the xOz plane (or $y = 0$ plane) $\mathbf{k} \rightarrow \mathbf{k}$ *b*) The matrix corresponding to the transformation A followed by B is $\mathbf{M}_C = \mathbf{M}_B\mathbf{M}_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ *ii*) *C* is mapping the vectors of the base as such $\mathbf{i} \rightarrow \mathbf{i}$ $\mathbf{j} \rightarrow \mathbf{k}$ This is a reflection in the plane $y = z$. $\mathbf{k} \rightarrow \mathbf{j}$	This question relied on the use of information given in the formulae booklet, and the majority of candidates coped very well with it. Many fell at the obvious hurdle in part (b), by supposing that "A followed by B" was represented by the matrix $\mathbf{M}_A\mathbf{M}_B$, rather than $\mathbf{M}_B\mathbf{M}_A$. Candidates who failed to attempt to describe the three transformations by a single, rather than multiple (ie composite), description were not awarded marks. There was no penalty for candidates who, for instance, called a plane a line, so long as they gave the right equation.

Question 5:

$a)$ Normal vectors to the plane are

$$\mathbf{n}_1 = \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} \; and \; \mathbf{n}_2 = \begin{pmatrix} 2 \\ 12 \\ -1 \end{pmatrix}$$

Let's work out the angle between these vectors

$$\mathbf{n}_1 . \mathbf{n}_2 = \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} . \begin{pmatrix} 2 \\ 12 \\ -1 \end{pmatrix} = 6 - 48 - 1 = -43$$

$$|\mathbf{n}_1| = \sqrt{9+16+1} = \sqrt{26} \; and \; |\mathbf{n}_2| = \sqrt{4+144+1} = \sqrt{149}$$

$$Cos\theta = \frac{\mathbf{n}_1 . \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{-43}{\sqrt{26}\sqrt{149}} \quad this \; gives \; \theta = 133.69^o$$

The acute angle between the plane is 180-133.69 $= 46.3^o$

$$b) \mathbf{r}. \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} = 2 \Leftrightarrow 3x - 4y + z = 2$$

$$\mathbf{r}. \begin{pmatrix} 2 \\ 12 \\ -1 \end{pmatrix} = 38 \Leftrightarrow 2x + 12y - z = 38$$

$c) \, i)$ A direction vector of the line of intersection is $\mathbf{u} = \mathbf{n}_1 \times \mathbf{n}_2$

$$\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 1 \\ 2 & 12 & -1 \end{vmatrix} = \begin{vmatrix} -4 & 1 \\ 12 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} \mathbf{k} + \begin{vmatrix} 3 & -4 \\ 2 & 12 \end{vmatrix} \mathbf{k}$$

$$\mathbf{u} = -8\mathbf{i} + 5\mathbf{k} + 44\mathbf{k}$$

A point belonging to both plane satisfies $\begin{cases} 3x - 4y + z = 2 \\ 2x + 12y - z = 38 \end{cases}$

$$\Leftrightarrow \begin{cases} 3x - 4y + z = 2 \\ 5x + 8y = 40 \end{cases} \quad for \; example \, (8, 0, -22)$$

Cartesian equations of the line of intersection is : $\dfrac{x-8}{-8} = \dfrac{y}{5} = \dfrac{z+22}{44}$

$$ii) |\mathbf{u}| = \sqrt{(-8)^2 + (5)^2 + (44)^2} = 45$$

The direction cosines are : $\dfrac{-8}{45}, \dfrac{5}{45} = \dfrac{1}{9}, \dfrac{44}{45}$

Exam report

This was the longest question on the paper, and contained a mixture of ideas from across the unit's topics. Most candidates coped very favourably with this variety and, minor arithmetical slips apart, marks were high. The biggest obstacle to success came in part (b)(i), where candidates needed to find the coordinates of any one point on the line of intersection. Setting *x*, *y* or *z* equal to zero and finding a value for the other two variables from the resulting simultaneous equations was the most common approach, and this worked well for most candidates.

Question 6:	Exam report
$a)$ Let's work out $\det(\mathbf{X}\text{-}\lambda\mathbf{I})=0$ $\Leftrightarrow \begin{vmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix} = 0 \Leftrightarrow (1-\lambda)(4-\lambda)-10=0$ $\Leftrightarrow \lambda^2-5\lambda-6=0$ $\Leftrightarrow (\lambda-6)(\lambda+1)=0$ The eigenvalues are $\lambda=6$ and $\lambda=-1$ $b)i)$ To find the eigenvectors, we solve $(\mathbf{X}\text{-}\lambda\mathbf{I})\begin{pmatrix} x \\ y \end{pmatrix}=0$ $For\ \lambda=-1, we\ have\ (\mathbf{X}+\mathbf{I})\begin{pmatrix} x \\ y \end{pmatrix}=0 \Leftrightarrow \begin{pmatrix} 2 & 2 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=0$ $\Leftrightarrow \begin{cases} 2x+2y=0 \\ 5x+5y=0 \end{cases} \Leftrightarrow x+y=0$ An eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $For\ \lambda=6, we\ have\ (\mathbf{X}-6\mathbf{I})\begin{pmatrix} x \\ y \end{pmatrix}=0 \Leftrightarrow \begin{pmatrix} -5 & 2 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=0$ $\Leftrightarrow \begin{cases} -5x+2y=0 \\ 5x-2y=0 \end{cases} \Leftrightarrow 5x=2y$ An eigenvector is $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ The matrices $\mathbf{D}=\begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}$ and $\mathbf{U}=\begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix}$ $ii)\ Det(\mathbf{U})=\begin{vmatrix} 1 & 2 \\ -1 & 5 \end{vmatrix}=5+2=7\ \ and\ \ \mathbf{U}^{-1}=\frac{1}{7}\begin{pmatrix} 5 & -2 \\ 1 & 1 \end{pmatrix}$ $iii)\ \mathbf{X}^5=\mathbf{U}\mathbf{D}^5\mathbf{U}^{-1}=\frac{1}{7}\begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} (-1)^5 & 0 \\ 0 & 6^5 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ 1 & 1 \end{pmatrix}$ $\mathbf{X}^5=\frac{1}{7}\begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} -5 & 2 \\ 6^5 & 6^5 \end{pmatrix}=\frac{1}{7}\begin{pmatrix} -5+2\times 6^5 & 2+2\times 6^5 \\ 5+5\times 6^5 & -2+5\times 6^5 \end{pmatrix}=\begin{pmatrix} 2221 & 2222 \\ 5555 & 5554 \end{pmatrix}$	This question was handled very confidently by most candidates, and apart from a small number of candidates who clearly deduced the matrix $\mathbf{X}^5$ from a calculator, despite the question's injunction to use the previous results, marks of 10, 11 or 12 were very common indeed.

Question 7:	Exam report
$a)\ \mathbf{M}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1&2\\-2&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-x+2y\\-2x+3y\end{pmatrix}=\begin{pmatrix}x'\\y'\end{pmatrix}$ All the points on the line are invariant so $x'=x$ *and* $y'=y$ this gives: $\begin{cases}x=-x+2y\\y=-2x+3y\end{cases}\Leftrightarrow\begin{cases}2x-2y=0\\2x-2y=0\end{cases}$ The line of invariant points has equation $y=x$. $b)\ \mathbf{M}\begin{pmatrix}x\\x+c\end{pmatrix}=\begin{pmatrix}-x+2(x+c)\\-2x+3(x+c)\end{pmatrix}=\begin{pmatrix}x+2c\\x+3c\end{pmatrix}=\begin{pmatrix}x'\\y'\end{pmatrix}$ It is obvious that $y'=x'+c$. For all c, the line $y=x+c$ is invariant. $c)\ \det(\mathbf{M})=\begin{vmatrix}-1&2\\-2&3\end{vmatrix}=-3+4=1$ Through a shear, the area is invariant. $d)$ The line with equation $y=-x$ has direction vector $\mathbf{u}_1=\begin{pmatrix}1\\-1\end{pmatrix}$ The image of the line $y=-x$: $\mathbf{M}\begin{pmatrix}x\\-x\end{pmatrix}=\begin{pmatrix}-1&2\\-2&3\end{pmatrix}\begin{pmatrix}x\\-x\end{pmatrix}=\begin{pmatrix}-3x\\-5x\end{pmatrix}=\begin{pmatrix}x'\\y'\end{pmatrix}$ $y'=-5x=\frac{5}{3}\times-3x=\frac{5}{3}x'.\qquad 5x'-3y'=0$ The image of the line $y=-x$ is the line $5x-3y=0$ A direction vector of this line is $\mathbf{u}_2=\begin{pmatrix}3\\5\end{pmatrix}$ Let's work out the angle between these lines $\mathbf{u}_1.\mathbf{u}_2=\begin{pmatrix}1\\-1\end{pmatrix}.\begin{pmatrix}3\\5\end{pmatrix}=3-5=-2\ ,\ \lvert\mathbf{u}_1\rvert=\sqrt{1+1}=\sqrt{2}$ *and* $\lvert\mathbf{u}_2\rvert=\sqrt{9+25}=\sqrt{34}$ $Cos\theta=\frac{\mathbf{u}_1.\mathbf{u}_2}{\lvert\mathbf{u}_1\rvert\lvert\mathbf{u}_2\rvert}=\frac{-2}{\sqrt{2}\sqrt{34}}\quad so\ \ \theta=104.04^o$ The acute angle between the lines is $180^o-104^o.04=76^o$ to the nearest degree.	This question was slightly tricky in some respects. However, the right approach cut through the apparent problems remarkably quickly, and many candidates were well up to the task. In part (a), it was essential to note that $x'=x$ and $y'=y$. in order to make rapid progress, although many candidates took the alternate route and found a single (repeated) eigenvalue of 1 for the matrix before proceeding successfully by that means. In part (b), it was important to work with $\mathbf{M}\begin{pmatrix}x\\x+c\end{pmatrix}$ from the outset, and then note that y'= x'+c also. In part (d), a similar start with $\mathbf{M}\begin{pmatrix}x\\-x\end{pmatrix}$ again cut out unnecessary work.

Question 8:	Exam report
$a)\ i)\ \det(\mathbf{P}) = \begin{vmatrix} 4 & -1 & 2 \\ 1 & 1 & 3 \\ -2 & 0 & a \end{vmatrix} = 4a + (a+6) + 2(0+2)$ $\det(\mathbf{P}) = 5a + 10$ $ii)\ \det(\mathbf{P})$ *when* $a = 3$ *is* $15 + 10 = 25$ $iii)$ $\mathbf{P}$ is singular when $\det(\mathbf{P})=0$. This happens when $a = -2$ $b)\ \mathbf{PQ} = 25\mathbf{I}$ $i)$ Multiply both sides by $\mathbf{P}^{-1}$: $\mathbf{P}^{-1}\mathbf{PQ} = 25\mathbf{P}^{-1}\mathbf{I}$ $\frac{1}{25}\mathbf{Q} = \mathbf{P}^{-1}$ $ii)\ (\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1} = \mathbf{Q}^{-1} \times \frac{1}{25}\mathbf{Q} = \frac{1}{25}\mathbf{I}$ $iii)\ \det(\mathbf{PQ}) = \det(\mathbf{P})\det(\mathbf{Q})$ $\det(25\mathbf{I}) = 25 \times \det(\mathbf{Q})$ $25^3 = 25\det(\mathbf{Q})$ *so* $\det(\mathbf{Q}) = 25^2 = 625$	This was the last question on the paper because, since it involves a 3 × 3 matrix, it was potentially the most time-consuming question. However, candidates found the structure of the question very helpful.

Grade boundaries							
Grade			A	B	C	D	E
Mark	Max 75		61	53	45	38	31