Question 1: Jan 2007 – Q8

The diagram shows the curve $y = \cos^{-1}x$ for $-1 \leq x \leq 1$.

(a) Write down the exact coordinates of the points $A$ and $B$. (2 marks)

(b) The equation $\cos^{-1}x = 3x + 1$ has only one root. Given that the root of this equation is $x$, show that $0.1 \leq x \leq 0.2$. (2 marks)

(c) Use the iteration $x_{n+1} = \frac{1}{3}(\cos^{-1}x_n - 1)$ with $x_1 = 0.1$ to find the values of $x_2$, $x_3$ and $x_4$, giving your answers to three decimal places. (3 marks)

Question 2: Jun 2007 – Q4

(b) The curve $y = 3^x$ intersects the line $y = x + 3$ at the point where $x = x$.

(i) Show that $x$ lies between 0.5 and 1.5. (2 marks)

(ii) Show that the equation $3^x = x + 3$ can be rearranged into the form

$$x = \frac{\ln(x+3)}{\ln 3}$$

(2 marks)

(iii) Use the iteration $x_{n+1} = \frac{\ln(x_n+3)}{\ln 3}$ with $x_1 = 0.5$ to find $x_3$ to two significant figures. (2 marks)

(iv) The sketch on Figure 1 shows part of the graphs of $y = \frac{\ln(x+3)}{\ln 3}$ and $y = x$, and the position of $x_1$.

On Figure 1, draw a cobweb or staircase diagram to show how convergence takes place, indicating the positions of $x_2$ and $x_3$ on the $x$-axis. (2 marks)
Question 3: Jan 2008 – Q3

The equation

\[ x + (1 + 3x)^{\frac{1}{3}} = 0 \]

has a single root, \( x \).

(a) Show that \( x \) lies between \(-0.33 \) and \(-0.32 \). \( (2 \text{ marks}) \)

(b) Show that the equation \( x + (1 + 3x)^{\frac{1}{3}} = 0 \) can be rearranged into the form

\[ x = \frac{1}{3}(x^4 - 1) \] \( (2 \text{ marks}) \)

(c) Use the iteration \( x_{n+1} = \frac{x_n^4 - 1}{3} \) with \( x_1 = -0.3 \) to find \( x_4 \), giving your answer to three significant figures. \( (3 \text{ marks}) \)

Question 4: Jun 2008 – Q3

A curve is defined for \( 0 \leq x \leq \frac{\pi}{4} \) by the equation \( y = x \cos 2x \), and is sketched below.

(a) Find \( \frac{dy}{dx} \). \( (2 \text{ marks}) \)

(b) The point \( A \), where \( x = \alpha \), on the curve is a stationary point.

(i) Show that \( 1 - 2x \tan 2x = 0 \). \( (2 \text{ marks}) \)

(ii) Show that \( 0.4 < \alpha < 0.5 \). \( (2 \text{ marks}) \)

(iii) Show that the equation \( 1 - 2x \tan 2x = 0 \) can be rearranged to become \( x = \frac{1}{2} \tan^{\frac{1}{2}} \left( \frac{1}{2x} \right) \). \( (1 \text{ mark}) \)

(iv) Use the iteration \( x_{n+1} = \frac{1}{2} \tan^{\frac{1}{2}} \left( \frac{1}{2x_n} \right) \) with \( x_1 = 0.4 \) to find \( x_3 \), giving your answer to two significant figures. \( (2 \text{ marks}) \)
The curve with equation \( y = x^3 + 5x - 4 \) intersects the \( x \)-axis at the point \( A \), where \( x = x \).

(a) Show that \( x \) lies between 0.5 and 1. \((2 \text{ marks})\)

(b) Show that the equation \( x^3 + 5x - 4 = 0 \) can be rearranged into the form \( x = \frac{1}{5}(4 - x^3) \). \((1 \text{ mark})\)

(c) Use the iteration \( x_{n+1} = \frac{1}{5}(4 - x_n^3) \) with \( x_1 = 0.5 \) to find \( x_3 \), giving your answer to three decimal places. \((2 \text{ marks})\)

(d) The sketch on Figure 1 shows parts of the graphs of \( y = \frac{1}{5}(4 - x^3) \) and \( y = x \), and the position of \( x_1 \).

On Figure 1, draw a cobweb or staircase diagram to show how convergence takes place, indicating the positions of \( x_2 \) and \( x_3 \) on the \( x \)-axis. \((2 \text{ marks})\)
Question 6: Jun 2009 – Q1

(a) The curve with equation

\[ y = \frac{\cos x}{2x + 1}, \quad x > -\frac{1}{2} \]

intersects the line \( y = \frac{1}{2} \) at the point where \( x = a \).

(i) Show that \( a \) lies between 0 and \( \frac{\pi}{2} \). \hspace{1cm} (2 marks)

(ii) Show that the equation \( \frac{\cos x}{2x + 1} = \frac{1}{2} \) can be rearranged into the form

\[ x = \cos x - \frac{1}{2} \] \hspace{1cm} (1 mark)

(iii) Use the iteration \( x_{n+1} = \cos x_n - \frac{1}{2} \) with \( x_1 = 0 \) to find \( x_3 \), giving your answer to three decimal places. \hspace{1cm} (2 marks)

Question 7: Jan 2010 – Q2

[Figure 1, printed on the insert, is provided for use in this question.]

(a) (i) Sketch the graph of \( y = \sin^{-1} x \), where \( y \) is in radians. State the coordinates of the end points of the graph. \hspace{1cm} (3 marks)

(ii) By drawing a suitable straight line on your sketch, show that the equation

\[ \sin^{-1} x = \frac{1}{4} x + 1 \]

has only one solution. \hspace{1cm} (2 marks)

(b) The root of the equation \( \sin^{-1} x = \frac{1}{4} x + 1 \) is \( \alpha \). Show that \( 0.5 < \alpha < 1 \). \hspace{1cm} (2 marks)

(c) The equation \( \sin^{-1} x = \frac{1}{4} x + 1 \) can be rewritten as \( x = \sin \left( \frac{1}{4} x + 1 \right) \).

(i) Use the iteration \( x_{n+1} = \sin \left( \frac{1}{4} x_n + 1 \right) \) with \( x_1 = 0.5 \) to find the values of \( x_2 \) and \( x_3 \), giving your answers to three decimal places. \hspace{1cm} (2 marks)

(ii) The sketch on Figure 1 shows parts of the graphs of \( y = \sin \left( \frac{1}{4} x + 1 \right) \) and \( y = x \), and the position of \( x_1 \).

On Figure 1, draw a cobweb or staircase diagram to show how convergence takes place, indicating the positions of \( x_2 \) and \( x_3 \) on the x-axis. \hspace{1cm} (2 marks)
Question 8: Jun 2010 – Q1

The curve \( y = 3^x \) intersects the curve \( y = 10 - x^3 \) at the point where \( x = a \).

(a) Show that \( a \) lies between 1 and 2.  \hspace{1cm} (2 marks)

(b) (i) Show that the equation \( 3^x = 10 - x^3 \) can be rearranged into the form \( x = \sqrt[3]{10 - 3^x} \). \hspace{1cm} (1 mark)

(ii) Use the iteration \( x_{n+1} = \sqrt[3]{10 - 3^{x_n}} \) with \( x_1 = 1 \) to find the values of \( x_2 \) and \( x_3 \), giving your answers to three decimal places. \hspace{1cm} (2 marks)
Iterative methods - exam questions - MS

## Question 1: Jan 2007 – Q8

(a) \( A(-1, n) \)

\[ B \left( \frac{0}{2}, \frac{\pi}{2} \right) \]

(b) \[ \cos^2 x - 3x - 1 = 0 \]

- \( f(0.1) = 0.17 \) allow 0.2, 0.1
- \( f(0.2) = -0.23 \) allow -0.2

Change of sign: root

\[ x_1 = 0.1 \]
\[ x_2 = 0.1569 = 0.157 \]
\[ x_3 = 0.1378 = 0.138 \]
\[ x_4 = 0.144 \]

Total 7

## Question 2: Jun 2007 – Q4

(i) \[ f(x) = 3^x - x - 3 \]

- \( f(0.5) = -1.77 \)
- \( f(1.5) = 0.696 \)

Change of sign: root M1A1 2

(ii) \[ 3^x = x + 3 \]

\[ \ln 3^x = \ln (x + 3) \]

\[ x = \ln (x + 3) \]

\[ x \approx 1.14 \] (A1)

\[ x_3 \approx 1.29 = 1.3 \]

M1 A1 2

## Question 3: Jan 2008 – Q3

(a) \[ (x + 1 + 3x)^\frac{1}{2} = 0 \]

- \( f(-0.32) = 0.1 \)
- \( f(0.33) = -0.01 \)

Change of sign: \(-0.33 < x < -0.32\)

\( x = -(1 + 3x)^\frac{1}{4} \)

\[ x^4 = 1 + 3x \]

\[ x^4 - 1 = x \]

M1 A1 2

(b) \[ x_1 = -0.3 \]

\( x_2 = -0.331 \) (AWRT)

\( x_3 = -0.329 \) (AWRT)

\( x_4 = -0.339 \)

A1 A1 3

Total 7

## Question 4: Jun 2008 – Q3

(a) \[ \frac{dy}{dx} = -x \sin 2x + x \cos 2x \]

\[ M1 \]

(b)(i) \[ 2 \cos 2x + \cos 2x = 0 \]

Change of sign: 0.4 < \( \alpha \) < 0.5

A1 2

(b)(ii) \[ f(0.4) = 0.2 \]

A1 2

(c) \[ 2 \tan 2x = \frac{1}{2x} \]

\[ x = \frac{1}{2} \tan^{-1} \left( \frac{1}{2x} \right) \]

M1 A1 2

## Question 5: Jan 2009 – Q3

(a) \[ f(x) = x^2 + 5x - 4 \]

\[ f(0.5) = -1.375 \]

Change of sign: 0.5 < \( \alpha \) < 1

M1 A1 2

(b) \[ x^2 + 5x - 4 = 0 \]

\[ 5x = 4 - x^2 \]

\[ x = \frac{1}{5} (4 - x^2) \]

M1 A1 2

(c) \( x_1 = 0.5 \)

\( x_2 = 0.775 \) (A1)

\( x_3 = 0.707 \)

A1 2

(d) \[ y \]

Total 7
Question 6: Jun 2009 – Q1

(i) \[ f(x) = \frac{\cos x}{2x+1} - \frac{1}{2} \]
\[ f(0) = \frac{1}{2}; \quad f\left(\frac{\pi}{2}\right) = -\frac{1}{2} \]
Change of sign \( 0 < \alpha < \frac{\pi}{2} \)  

(ii) \[
\begin{align*}
\frac{\cos x}{2x+1} &= \frac{1}{2} \\
2\cos x &= 2x + 1 \\
2\cos x - 1 &= 2x \\
or, \quad \cos x &= x + \frac{1}{2}
\end{align*}
\]
\[ x = \cos x - \frac{1}{2} \]

\[ x_1 = 0 \]
\[ x_2 = 0.5 \]
\[ x_3 = 0.378 \]

Question 7: Jan 2010 – Q2

(i) \[ x_2 = 0.902 \]
\[ x_3 = 0.941 \]

(ii) \[
\begin{tikzpicture}
\draw[->] (-2,0) -- (4,0) node[right] {$x$};
\draw[->] (0,-2) -- (0,4) node[above] {$y$};
\draw[thick] (-2,-2) -- (4,4);
\draw[thick] (-2,0) -- (0,0);
\draw[thick] (0,-2) -- (0,0);
\draw[thick] (2,-2) -- (2,2);
\draw[thick] (2,0) -- (2,2);
\draw[thick] (4,-2) -- (4,4);
\draw[thick] (4,0) -- (4,4);
\end{tikzpicture}
\]

Question 8: Jun 2010 – Q1

1(a) \[ f(x) = 3^x - 10 + x^2 \] (or reverse)
\[ f(1) = -6 \]
\[ f(2) = 7 \]
Change of sign \( \therefore 1 < \alpha < 2 \)

OR

\[
\begin{align*}
\text{LHS} (1) &= 3 \\
\text{RHS} (1) &= 9 \\
\text{LHS} (2) &= 9 \\
\text{RHS} (2) &= 2 \\
\text{At 1} \quad \text{LHS} < \text{RHS}, \quad \text{At 2} \quad \text{LHS} > \text{RHS} \quad \therefore 1 < \alpha < 2
\end{align*}
\]

(b)(i) \[ x^3 = 10 - x^3 \]
\[ x^3 = 10 - 3^x \]
\[ x = \sqrt[3]{10 - 3^x} \]

(b)(ii) \[ x_1 = 1.913 \]
\[ x_2 = 1.221 \]

Total 5